Interpolation/Extrapolation of Frequency Domain Responses Using the Hilbert Transform

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Abstract— **The Hilbert transform relates the real and the imaginary parts of the transfer function of a causal system. The objective of this paper is to illustrate how the Hilbert transform relationship can be utilized to interpolate/extrapolate measured frequency domain responses of devices. Sample numerical examples are presented to illustrate the efficacy of this method.**

I. Introduction

SYSTEM measurements in the time domain are easier to perform since the waveforms of interest are all real. However, one disadvantage of performing measurements in the time domain is limited dynamic range. Frequency domain measurement equipment benefits from large dynamic range. Furthermore, frequency domain measurements may be carried out either over an entire range of frequencies or selectively over a band of frequencies. Theoretically, it is possible to extract a time domain response from these measurements by an inverse Fourier transform. But, if the measurements are made in a noisy environment, or over a selected band of frequencies, it is difficult to recover the entire time domain response.

The time domain response of a physical system is always causal, since the signal is nonzero only after a certain interval of time. However, since band-limited complex frequency domain data does not guarantee causality in the time domain, nor a real time domain response, measurements carried out in the frequency domain do not truly represent the transient response of the system. Even so, we establish that it is possible to extract a causal response by interpolating the complex frequency domain data under the premise that the time domain signal must be causal. We use the principle of causality to extrapolate/interpolate frequency domain response [1].

In general, the real and the imaginary parts of the complex frequency domain data are independent of each other. However, the causality of the time domain signal, denoted as $h(t)$, assures us that the real and imaginary components of the frequency domain are related through the Hilbert transform. If we denote $H_{\Re}(j\omega)$ as the real part and $H_{\Im}(j\omega)$ as the imaginary part of the transfer function, $H(j\omega)$, obtained from the Fourier transform of $h(t)$, then, from causality, they have to be related by the Hilbert transform [1]–[9]. The physical principal of causality imposed some constraints on the real and the imaginary parts of the transfer functions. The relationship was originally developed by Kramers and Kronig [2]–[4]. James and Andrasic [5] have used this approach to minimize the effects of noise on experimental data. Arabi *et al.* [6] has used the Hilbert transform technique to generate causal time domain responses of multiconductor transmission lines by enforcing the Kramers–Kronig relationship between the dielectric constant and the loss tangent of any dielectric material. Tesche has used this technique [7] and [8] to generate a causal time domain response from bandlimited frequency domain data. The property that the real and the imaginary parts of the frequency domain data correspond to the even and odd parts of $h(t)$ is exploited in extracting a causal response from complex band-limited frequency domain data.

Since we process discrete frequency domain data, we handle frequency and time domain signals in the form of sequences. Numerical results are presented to demonstrate the utility of this technique.

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II. Interpolation/Extrapolation of Frequency Domain Data

A technique to extrapolate/interpolate data in the frequency domain utilizing the Fourier Transform to implement the Hilbert transform is described. Before the algorithm is described, it is useful to know something about the available frequency domain data. Assume that we have a complex frequency domain data between frequencies f_1 and f_4. Consider a missing band between f_2 and f_3. The frequency domain data is sampled at $(n_2\text{-}n_1)$ frequency points between f_2 and f_1, and at $(n_4\text{-}n_3)$ points between f_4 and f_3. This is expressed as a vector

$$H[n_1 : n_4] = [H_{n1} \cdots H_{n2},\, 0 \cdots 0,\, H_{n3} \cdots H_{n4}] \quad (1)$$

It is now our objective to interpolate this missing data between n_2 and n_3. As a first step:

1) The available bandlimited frequency domain data is padded with zeros to ensure a length of n points where n is given by $N/2 + 1$, and N is $[2, 4, 8, \cdots, 1024, 2048, \cdots]$, providing a sequence of even length. The complex data is now given by

$$H[1 : n] = H[1 : N/2 + 1]$$
$$= [0, 0, \cdots, H_{n1}, \cdots, H_{n2}, 0, 0, \cdots, 0,$$

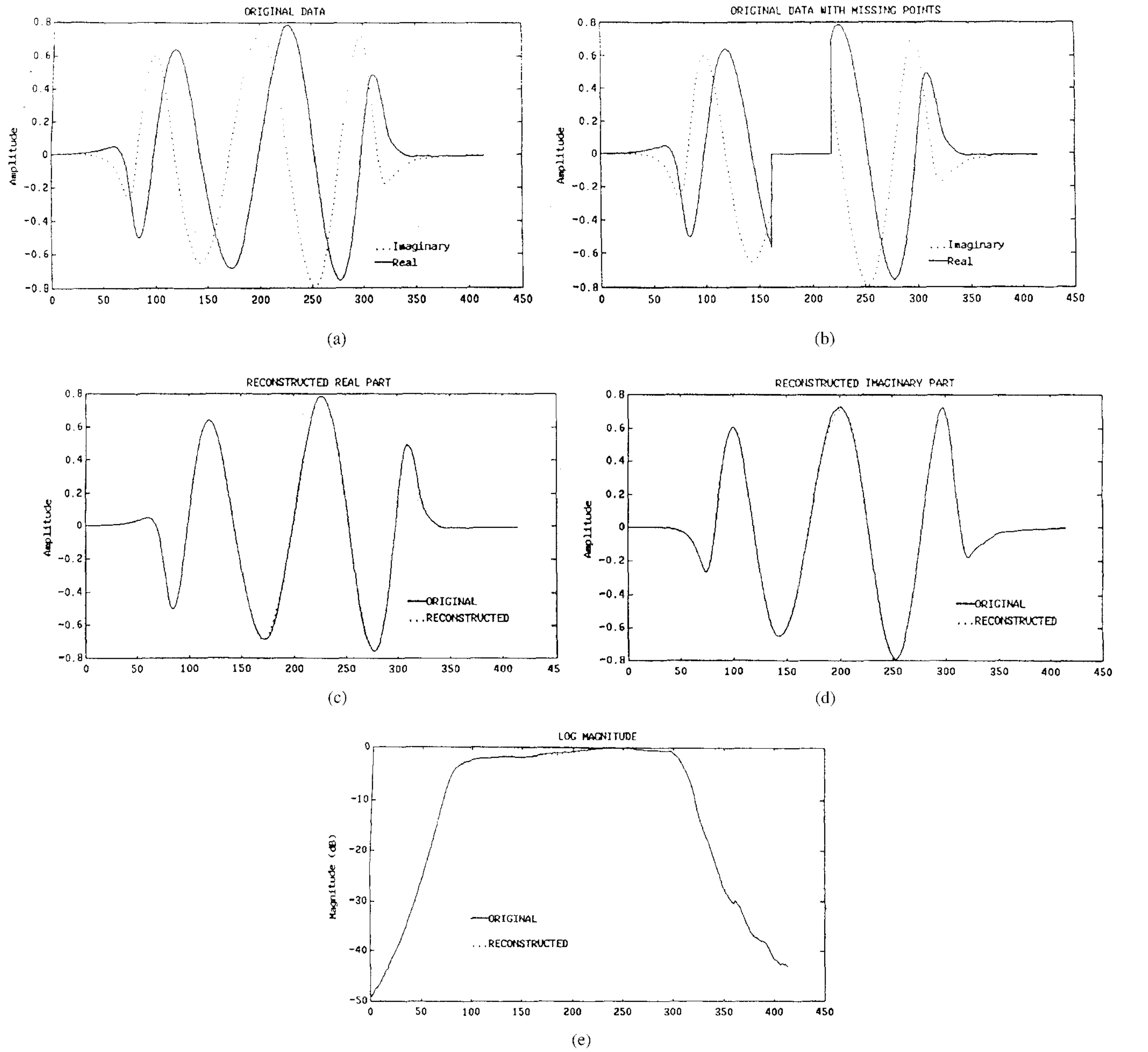

Fig. 1. These are plots of the frequency domain data of a microstrip band-pass filter [Interpolation results]. (a) Plot of the real and imaginary parts of the original data. (b) Plot of the real and imaginary parts of the original data showing the missing band. (c) Plot of reconstructed real part of the original real part. (d) Plot of reconstructed imaginary part of the original imaginary part. (e) Plot of log-magnitude of both the original and the reconstructed data.

$$H_{n3}, \cdots, H_{n4}, 0, 0, \cdots, 0]. \quad (2)$$

2) This complex sequence is altered to obtain a complex consequence of length N. This is done by appending the complex conjugate of the sequence to the original data

$$H[1:N] = [H[1:N/2+1], \overline{H}[N/2:2]]. \quad (3)$$

3) The complex sequence is now split into its real and imaginary parts

$$H_R = \text{Real}\,[H] \quad (4)$$

$$H_I = \text{Imag}\,[H]. \quad (5)$$

4) An inverse discrete Fourier transform of H_R results in an even sequence $h_e[n]$

$$h_e(1:N) = \text{Real}\,[\text{IFFT}(H_R)] \quad (6)$$

and

$$h_e[n] = h_e[-n]. \quad (7)$$

This is in fact the even part of the time domain sequence. The numerical implementation and the properties of the Hilbert transform may be found elsewhere [11].

5) Before proceeding further, it is important to know that there are sharp discontinuities in the frequency domain signal. In order to deal with this situation, we will

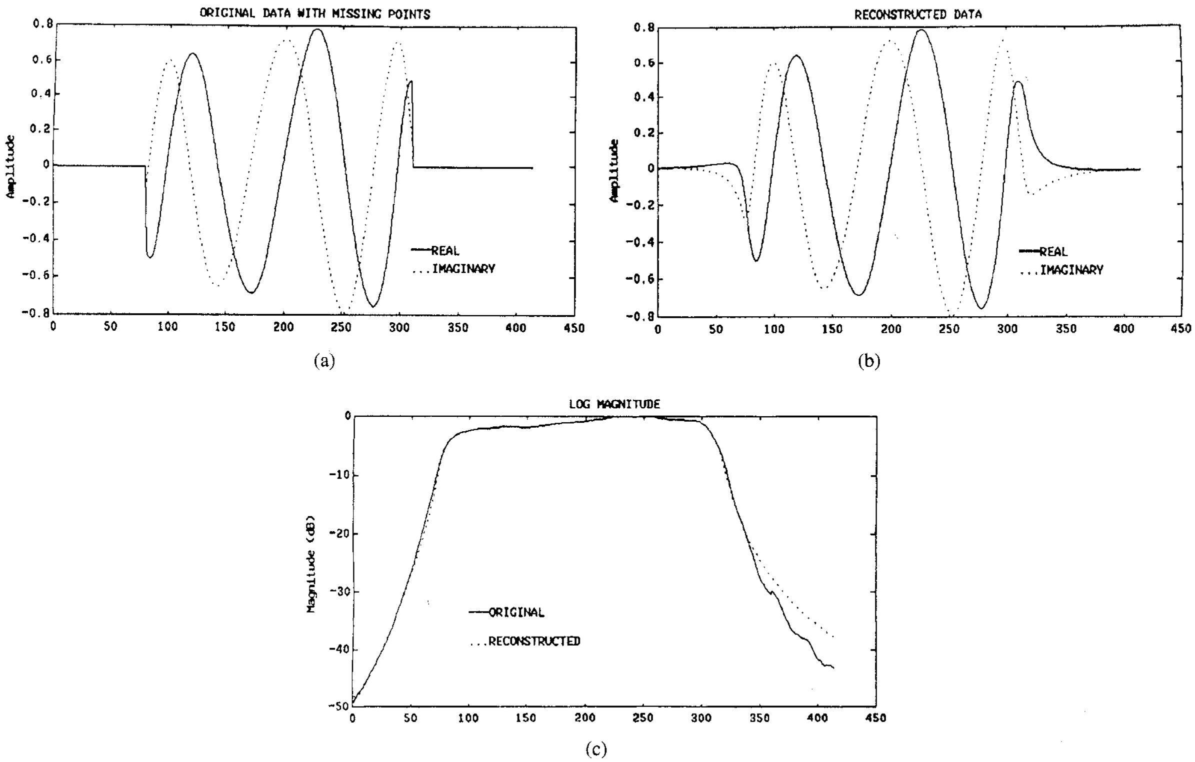

Fig. 2. These are plots of the frequency domain data of a microstrip band-pass filter. [Extrapolation results]. (a) Plot of the real and imaginary parts of the original data showing the missing band. (b) Plot of real and imaginary parts of the reconstructed data. (c) Plot of log-magnitude of both the original and the reconstructed data.

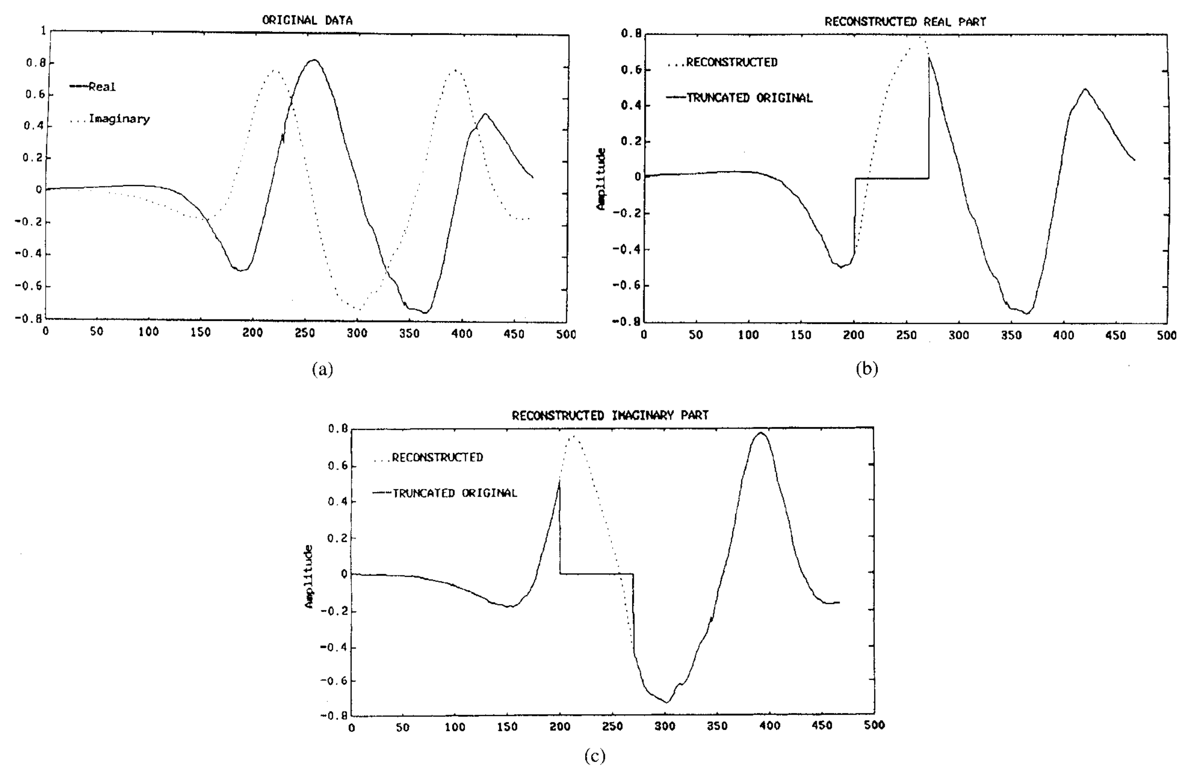

Fig. 3. These are plots of the frequency domain data of another microstrip filter [Interpolation of a considerably large number of missing points]. (a) Plot of the real and imaginary parts of the original data. (b) Plot of reconstructed real part and the original real part showing the missing points. (c) Plot of reconstructed imaginary part of the original imaginary part showing the missing points.

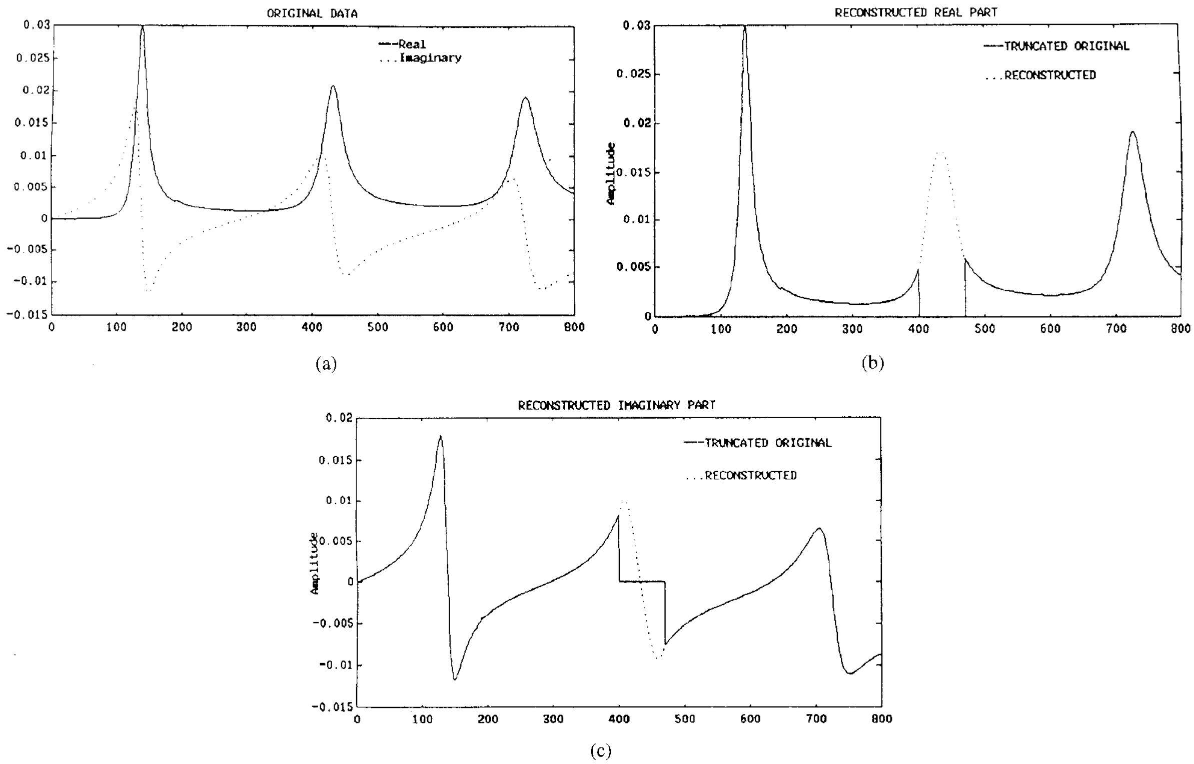

Fig. 4. These are plots of the frequency domain data for the input impedance of a dipole antenna. (a) Plot of the real and imaginary parts of the original data. (b) Plot of reconstructed real part and the original real part showing the missing points. (c) Plot of reconstructed imaginary part and the original imaginary part showing the missing points.

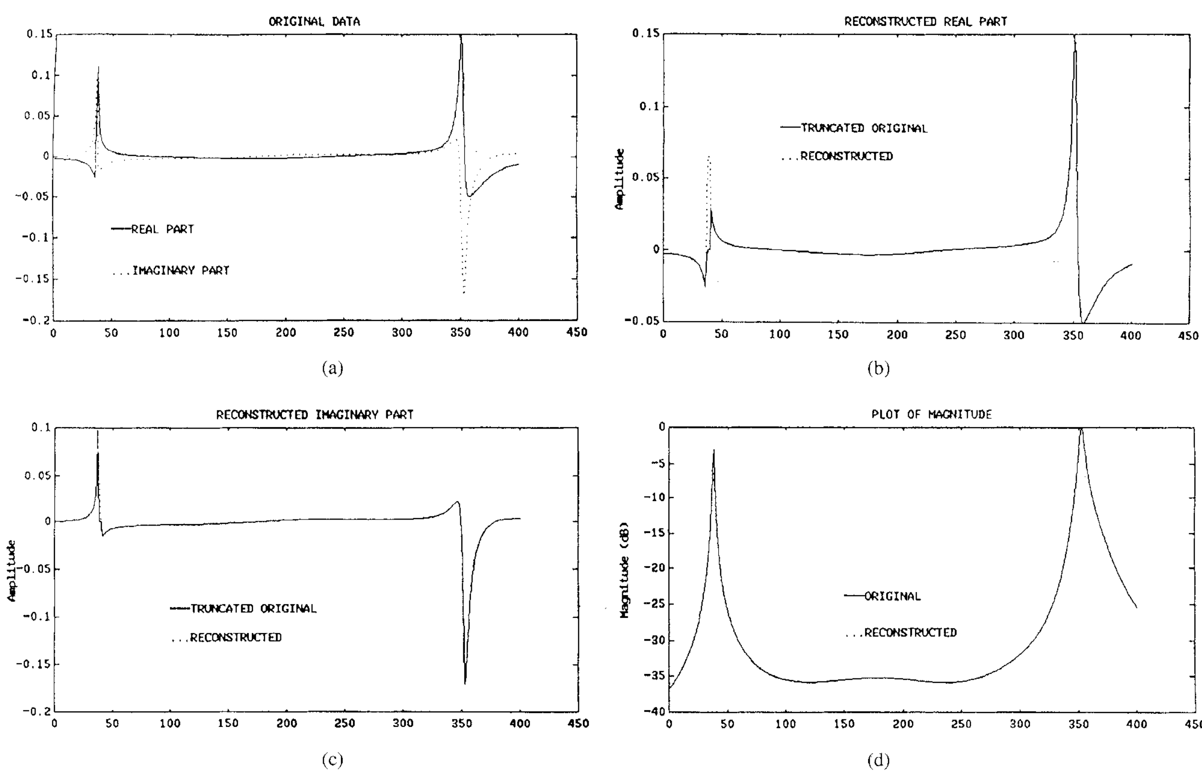

Fig. 5. These are plots of the frequency domain data for a microstrip notch filter. (a) Plot of the real and imaginary parts of the original data. (b) Plot of reconstructed real part and the original real part showing the missing points. (c) Plot of reconstructed imaginary part and the original imaginary part showing the missing points. (d) Plot of log-magnitude of both the original and the reconstructed data.

have to multiply the time domain sequence with a window.

A Hanning window of length N is multiplied with the time domain sequence. The resulting frequency domain sequence will now be filtered or "smoothed" [10].

The Hanning window is given by

$$W(n) = \begin{Bmatrix} 0.5 - \dfrac{0.5\cos(2\pi n)}{N} & 0 \le n < N \\ 0 & \text{otherwise} \end{Bmatrix}. \quad (8)$$

Hence

$$h_e(1:N) = h_e(1:N) * W(1:N) \quad (9)$$

where the $*$ denotes the convolution.

6) The odd sequence is obtained from the even sequence by making use of the relationships available in [11].

We have

$$h_o(1:N) = [0 \;\; h_e(2:N/2) \;\; 0 \;\; -h_e(N/2+2:N)] \quad (10)$$

and

$$h_o[n] = -h_o[-n]. \quad (11)$$

7) The discrete Fourier transform of this odd sequence will give the imaginary part of the spectrum as stated earlier

$$H_I^{new} = \text{Imag}\,[\text{FFT}(h_o)]. \quad (12)$$

8) A substitution for the missing points is made in the imaginary part of the original sequence using the sequence obtained in Step 7)

$$\begin{aligned} H_I^{Sub} = [&H_I^{new}(1:n_1-1), \\ &H_I(n_1:n_2), \\ &H_I^{new}(n_2+1:n_3-1), \\ &H_I(n_3:n_4), \\ &H_I^{new}(n_4+1:N/2+1)]. \end{aligned} \quad (13)$$

9) This sequence is copied to obtain a sequence of length N which is an improved version of the original sequence H_I

$$H_I^{Sub} = [H_I^{Sub}[1:N/2+1], -H_I^{Sub}[N/2:2]]. \quad (14)$$

10) The inverse discrete Fourier transform of this sequence will give us the odd sequence again

$$h_o^{new} = \text{IFFT}[jH_I^{Subs}] \quad (15)$$

since

$$h[n] = h_e[n] + h_o[n]. \quad (16)$$

11) We get the modified version of h_e, from

$$\begin{aligned} h_e^{new} = [&h_e(1), h_o^{new}(2:n/2), h_e(N/2+1), \\ &-h_o^{new}(N/2+2:N)]. \end{aligned} \quad (17)$$

12) The discrete Fourier transform of this sequence obtained in the previous step will give us the real part of the spectrum as stated earlier

$$H_R^{new} = \text{Real}\,[\text{FFT}(h_e^{new})]. \quad (18)$$

13) A substitution for the missing points is made in the Real part of the original sequence using the sequence obtained in Step 12), as

$$\begin{aligned} H_R^{Sub}[&H_R^{new}(1:n_1-1), \\ &H_R(n_1:n_2), \\ &H_R^{new}(n_2+1:n_3-1), \\ &H_R(n_3:n_4), \\ &H_R^{new}(n_4+1:N/2+1)]. \end{aligned} \quad (19)$$

14) This sequence is copied to obtain a sequence of length N

$$\begin{aligned} H_R^{Subs} = [&H_R^{Sub}[1:N/2+1], \\ &H_R^{Sub}[N/2:2]] \end{aligned} \quad (20)$$

which is an improved version of the original sequence H_R.

15) The resulting sequence is subject to an inverse discrete Fourier transform to obtain the even sequence

$$h_e(1:N) = \text{Real}\,[\text{IFFT}\,(H_R^{Subs})]. \quad (21)$$

16) As in Step 5), this time domain sequence is multiplied with the Hanning window.

17) Subsequent signal processing are iterations of Steps 6)–16).

The above procedure will interpolate the missing band of frequencies. The reconstructed sequence will now be the complex sequence given by

$$H^{Rec}[1:n_4] = H_R^{Subs}[1:n_4] + jH_I^{Subs}[1:n_4] \quad (22)$$

and by comparing with (1) we have

$$\begin{aligned} &H^{Rec}[1:n_4] = \\ &[H_{n1}, \cdots, H_{n2}, \cdots, H_{n2+1}^{Rec}, \cdots, H_{n3-1}^{Rec}, H_{n3}, \cdots, H_{n4}]. \end{aligned} \quad (23)$$

It is worthwhile to note that by making use of the Hanning window, although we have overcome the difficulties due to discontinuities at the ends of the missing band, we might suffer a loss of resolution. This is not a serious problem and its effects can be minimized as shown in the numerical examples.

III. Numerical Results

As a first example consider the frequency domain data of a microstrip filter measured using the HP 8510B Network Analyzer. The device is a band-pass filter and its characteristics are measured at 415 points from 4.2069–8.5 GHz as shown in Fig. 1(a). Since in this example, the final result of interest is extrapolation/interpolation of the data in the frequency domain, translating the frequency axis by equating 4.2069 to 0 GHz does not really affect the results. In this example, we throw away the data points from 161–219 which corresponds to the frequency points of 5.4875–6.2375 GHz, as shown in Fig. 1(b). The missing data points are replaced by zeros, and the data is padded by zeros from 416–1025 sample points. The objective is to interpolate the missing data values by utilizing the principles of Section II.

Fig. 1(c) describes the interpolated data points utilizing the iterative principles described in Section IV to interpolate samples 161–219. In Fig. 1(c) and (d), the reconstructed data is compared to that of the original data, both in the real and in the imaginary parts, respectively. Fig. 1(e) plots the log-magnitude plot of the bandpass filter with both the real data and the reconstructed data superimposed. So for this example, the objective has been to interpolate part of the pass band response from stop band data.

For the second example, we try to extrapolate the stop band data from the pass band response. Again Fig. 1(a) displays the 415 point band-pass filter data. Out of the 415 points, data from 1–80 and 310–415 points are discarded. These are the 3 dB points of the filter. This is equivalent to discarding the data from 3.5–4.9875 GHz and 6.5875–8.5 GHz. This is shown in Fig. 2(a). The extrapolated data is generated by utilizing the Hilbert transform iteration, described in the previous section. The extrapolated data matched well with the original data as illustrated by Fig. 2(b) and (c). It was difficult to match the out of band response below 30 dB, because the 50 Ω matched loads used in our experiments had a S_{11} value, which did not go below 30 dB.

For the third example consider the measured data of a microstrip filter measured between 4.2069–8.0013 GHz using 468 points. The data is shown in Fig. 3(a). We now remove a large number of data points in the pass band from 201–270. The Hilbert transform technique was used to fill in the missing data points producing interpolated responses for the real and imaginary parts of the data as shown in Fig. 3(b) and (c). The interpolated data agrees well with the original data shown in Fig. 3(a).

For the fourth example consider the interpolation of input impedance of a dipole antenna. The antenna is considered to be 2 m ($= L$) long and of radius 0.1 mm ($-R$). The input impedance of the center fed dipole was computed at every 1 MHz interval up to 800 MHz and 801 data samples are considered. The data measured was generated utilizing the commercially available code AWAS [12]. The original data is shown in Fig. 4(a). Next we excise data from 401–470 MHz which is equivalent to removing a peak in the real part and a fraction of the peak in the input reactance of the imaginary part. Next the Hilbert transform relationship is utilized to interpolate the input impedance of the dipole antenna in the missing band. The interpolated data are shown in Fig. 4(b) and (c). Even though the peak is positioned correctly, the amplitudes are underestimated. It has been observed that for thick dipole antennas (where the L/R ratio is small) the peak is reproduced more accurately than for the thin dipole antennas. The interpolated results more accurately match the actual data, since for an antenna with small L/R, the peaks in the impedances are wider and the FFT becomes much more well behaved.

As the final example, let us consider the measured data of a microstrip notch filter between the frequencies 2.0–6.0 GHz. Fig. 5(a) shows the original data with real and imaginary parts. In this case, most of the first peak is removed, i.e., data points from 35–41. Fig. 5(b) and (c) shows the reconstructed real and imaginary parts respectively, while Fig. 5(d) shows the plot of the log-magnitude. The reconstructed data generated from the methodology described in Section II closely matches with the original data.

IV. CONCLUSION

Currently work is underway to find out the regions of validity of this approach and when it breaks down. Finally, what is the minimum number of effective bits required in the data to successfully perform such data interpolation/extrapolation. Solution of these important problems will further enhance the potential of this method.

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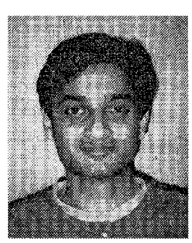

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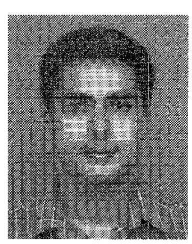

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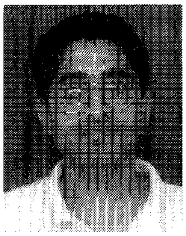

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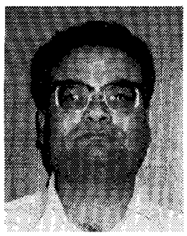

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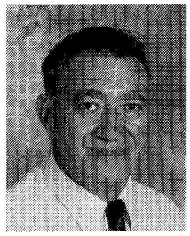

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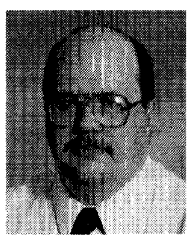
Dr. Vannicola has participated in studies on new trends in signal processing for radar, communication systems, speech processing, and artificial intelligence. He has served on many committees within the Department of Defense dealing with polarimetric processing, adaptive processing, electron devices, and artificial intelligence.



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