Using the Bhattacharyya Parameter for Design and Analysis of Cooperative Wireless Systems

Josephine P. K. Chu, *Student Member, IEEE,* Raviraj S. Adve, *Senior Member, IEEE,* and Andrew W. Eckford, *Member, IEEE*

***Abstract*—A simplified method of analysis and design based on the Bhattacharyya parameter (BP) in conjunction with the union bound and weight enumeration is presented for relay channels using coded cooperation. This method is particularly suitable for low-complexity relay systems employing demodulate-and-forward, focusing on the problems of relay selection and outage analysis. These applications are chosen to illustrate the use of the BP in scenarios where analytical solutions are otherwise unattainable. In terms of relay selection, it is shown that BP-based relay selection has essentially the same performance as density evolution, though with much lower complexity. It is further shown that BP-based relay selection can be applied to fractional cooperation, where each relay only forwards a fraction of the source codeword. In terms of analysis, it is shown that weight enumeration with BP can be used to provide a close approximate to the upper bound on the outage probability of fractional cooperation, again with much lower computational complexity than density evolution.**

***Index Terms*—Communication systems, channel coding, relays, cooperative systems.**

I. INTRODUCTION

IN wireless systems, relays can be used to combat the detrimental effects of fading channels. A typical three-node relay channel consists of a source node (which generates data), a destination node (which receives the data), and a relay node (which assists the source in communicating its message to the destination). Relay channels were studied in depth in [1], where the upper bound on the capacity of the relay channel was found, and achievable rates of decode-and-forward (DF) and compress-and-forward (CF) relaying were presented. Recently there has been a renewed interest in relay channels due to the technological advances over the years, potentially allowing for the use of relays in mesh and sensor networks. There are published papers analyzing the achievable rates of DF, CF and/or amplify-and-forward (AF) in both the small three-node networks and large multi-node networks, where the channels can be full-duplex or half-duplex, such as [2]–[4]. The diversity-multiplexing tradeoff of DF and AF for orthogonal and non-orthogonal channels were presented in [5] and [6] respectively. There are also studies on coding strategies realizing DF in relay channels (see, e.g., [7], [8] and the references therein). Innovative coding schemes that combine the benefits of DF and AF are also available. Examples include [9], [10], where the soft information from decoding the source signal is used to form soft signal at the relay, which is then transmitted to the destination node.

In this paper, we are motivated by distributed wireless networks with complexity-constrained hardware, such as sensor networks, for which *demodulate-and-forward* (DemF) [11], [12] may be the appropriate relaying strategy. In DemF, instead of decoding the received signal, the relay *demodulates* it, and transmits the demodulated bits (or parity bits formed from the demodulated bits) to the destination. This concept was implemented in conjunction with coded cooperation in [15] using low-density generator matrix (LDGM) [13] and repeat-accumulate (RA) [14] codes. The work in [15] showed that full diversity is obtained with these coding schemes, even though only simple operations are performed at the relay. Meanwhile, *fractional cooperation*, where each relay only forwards a fraction of the source codeword, was introduced in [15], [16]. Fractional cooperation is a flexible scheme, as it allows the burden of relaying to be distributed among several nodes in the network. It was shown that with fractional cooperation, as long as the number of relays used is greater than a threshold value, r_c, each additional relay increases the diversity order by one [15].

In analyzing and/or designing networks based on cooperation via error control coding (as opposed to, say, AF) a serious drawback is that closed-form analysis of coded cooperation is generally unavailable. The published work is based largely on time-consuming simulations or analysis via density evolution (DE) [17]. However, DE is an asymptotic concept based on the assumption of large blocklengths. Thus, DE does not always account for the peculiarities of the finite-length code being used. In this paper we propose the use of the *Bhattacharyya parameter* (BP) [18], in conjunction with the weight enumerator (WE) of the source's codeword, as a system analysis and design tool. The BP provides a measure of the quality of the effective channel, and is coupled with a WE that incorporates the specifics of the code. Using the two together (called BP or the BP approach here for convenience) provides a bound on the frame error rate (FER); one can therefore use the BP for analysis, e.g., to determine a performance measure such as outage probability.

In this paper we use the BP approach to address two specific problems found to be particularly intractable in previous work [15], [16], [19]: relay selection in large-scale networks,

Submitted to IEEE TRANSACTIONS ON WIRELESS COMMUNICATIONS, January 17, 2008. Material in this paper was presented in part at the IEEE International Conference on Communications, Beijing, China, May 2008.

Josephine P. K. Chu and Raviraj S. Adve are with The Edward S. Rogers Sr. Department of Electrical and Computer Engineering, University of Toronto, 10 King's College Road, Toronto, Ontario, Canada M5S 3G4. Emails: {chuj,rsadve}@comm.utoronto.ca

Andrew W. Eckford is with the Department of Computer Science and Engineering, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3. Email: aeckford@yorku.ca

and determination of the outage probability of fractional cooperation. However, we emphasize that the applications of the BP go beyond just the DemF applications explored here. Some examples include the design of finite-length codes in various scenarios, and the analysis of a hybrid DemF/DF scheme, wherein a relay may attempt some partial decoding to get a better estimate of the source's message.

Relay selection is motivated by the fact that, when multiple potential relays are available, it is impractical, and possibly undesirable, for all relays to try to help the source. This is because increasing the number of relays leads to an increase in total transmission power overhead, in addition to a decrease in transmission rate if orthogonal channels are used. *Relay selection*, where a single best relay is selected out of the pool of available relays, achieves full diversity order without the sacrifices in transmission rate and energy overhead. Relay selection also has the distinct advantage over distributed space-time codes of not requiring synchronization. Relay selection for DF [20], [21] and AF [22] have well-established selection criteria. For systems where coded cooperation is used (including as an implementation of DF), DE can be used to decide which relay provides the best error performance. However, even though DE is more efficient than simulations, and can be performed off-line with the data stored in a lookup table, the FER must be stored for every possible realization of all the channels. In [19], a simple heuristic using mutual information was introduced to find the optimal relay, but in that case the state of every channel in the relay network must be known. Under this scheme, the source-destination (S-D) channel must be communicated to all the relays, significantly increasing the communication overhead.

This paper considers relay selection using the BP when error performance is used as the selection criterion. In conjunction with the weight enumeration of the codewords, the use of the BP provides a significantly simpler approach to relay selection. Only the source-relay (S-R) and relay-destination (R-D) channel state information is required, allowing calculation to be performed at each relay (without knowledge of the S-D channel strength). This, in turn, minimizes the transmissions needed to communicate the channel coefficients. As the results of simulations will show, the error performance of relay selection via the BP is essentially indistinguishable from exhaustive search. Relay selection can also be applied to fractional cooperation, where the r_b best relays are chosen out of a pool of r_a available relays. Let $\tilde{r}_c(r_a)$ be the number of relays required to achieve diversity order of 2, given the relaying fraction, relaying scheme, and encoding and decoding schemes. We will show that with r_a relays available to assist the source, as long as $r_b \geq \tilde{r}_c(r_a)$, the diversity order of the system is equal to $r_a - \tilde{r}_c(r_a) + 2$.

In addition to relay selection, the BP approach can also be used as an efficient method to obtain a close approximation to the upper bound on the outage probability of fractional cooperation. A Gaussian distribution is used to approximate the effective channel signal after decoding "turbo-like" error correcting codes [23], which allows quick analysis and design of those codes. Using the BP to study outage probability is extremely flexible, as the effects of changing the average channel conditions, the fraction of source codeword relayed by each relay, and the number of relays can be obtained easily. Finally, we use the BP to illustrate the effects of *limited decoding* [24] at a relay. Limited decoding allows either a limited number of iterations of the sum-product algorithm (SPA) [25], or simpler decoding algorithm, such as the Gallager A algorithm [26], to be used at the relay. In this case, the effective S-R channel is improved. Clearly, determining how each iteration improves the overall system performance is analytically complicated; but fairly straightforward using the BP. Note that the limited decoding used in this paper is different than that suggested in [9], [10]; in our implementation, the information regarding the S-R channel is not embedded in the transmitted data, and the decoded data is quantized before being transmitted.

This paper is organized as follows: In Section II we present background information, including the system model, information on DemF cooperative coding, fractional cooperation, and the use of the BP. In Section III we illustrate how the BP approach can be used for relay selection and for analysis of the outage probability of relay channels. Simulation results for the various applications of the BP approach are presented in Section IV. Finally, some concluding remarks are drawn in Section V.

II. System Model and Background

This section presents the system model under consideration and a brief background on the concepts used extensively in this paper: DemF and fractional cooperation. A brief overview of the union bound and the BP is also presented.

A. System Model

The relay channel consists of a source (S) node, a destination (D) node and a total of r relays (R). A quasi-static Rayleigh fading channel model is used to describe the links between the nodes. It is assumed that all receivers have accurate channel state information, and the S-R channel signal-to-noise ratio (SNR) are known at the destination as well. Also, it is assumed that the channels between the nodes are half-duplex, and due to the fact that symbol synchronization is not available, orthogonal channels are used to facilitate transmissions between various nodes. These assumptions are frequently invoked for low-complexity wireless hardware [27], [28].

Transmission is separated into two phases. In the first phase, the source node forms a codeword $\mathbf{d}_S \in \{0,1\}^{n_S}$ of rate R_S. Let $\xi : \{0,1\}^{n_S} \to \{+1,-1\}^{n_S}$ represent a *modulation function*, mapping 0 to $+1$ and 1 to -1, respectively (with $\xi^{-1}(\cdot)$ as the inverse operation). Assuming that binary phase-shift keying (BPSK) is used, the codeword $\mathbf{d}_S$ is mapped to $\mathbf{c}_S \in \{+1,-1\}^{n_S}$ using the function $\xi(\cdot)$. The source node then *broadcasts* the binary codeword $\mathbf{c}_S$, and the discrete-time received signals at relay i and D are

$$\mathbf{y}_{\mathrm{SR},i} = h_{\mathrm{SR},i}\mathbf{c}_S + \mathbf{n}_{R,i}, \qquad i \in \{1,\ldots,r\} \tag{1}$$

$$\mathbf{y}_{\mathrm{SD}} = h_{\mathrm{SD}}\mathbf{c}_S + \mathbf{n}_{\mathrm{SD}}, \tag{2}$$

where $h_{\mathrm{SR},i}$ and h_{SD} are the channel coefficients between S and relay i and between S and D respectively, and $\mathbf{n}_{R,i}$ and

$\mathbf{n}_{\mathrm{SD}}$ are independent additive white Gaussian noise (AWGN) with variance $N_{0,R,i}$ and $N_{0,D}$ respectively. After the first transmission phase, relay i processes the received signal $\mathbf{y}_{\mathrm{SR},i}$, as described in the next section. It forms a new codeword $\mathbf{d}_{R,i}$ based on the received signal and generates $\mathbf{c}_{R,i} = \xi(\mathbf{d}_{R,i})$. In the second phase, each relay transmits the codeword, and the discrete time received signal at D is

$$\mathbf{y}_{\mathrm{RD},i} = h_{\mathrm{RD},i}\mathbf{c}_{R,i} + \mathbf{n}_{\mathrm{RD},i}, \tag{3}$$

where $h_{\mathrm{RD},i}$ is channel coefficient from relay i to D and $\mathbf{n}_{\mathrm{RD},i}$ is the AWGN with variance $N_{0,D}$. The average received SNR of the channel between S and relay i is

$$\bar{\gamma}_{\mathrm{SR},i} = \mathbb{E}[\gamma_{\mathrm{SR},i}] = \mathbb{E}[|h_{\mathrm{SR},i}|^2]/N_{0,R,i}, \tag{4}$$

where $\mathbb{E}[\cdot]$ is the statistical expectation function and $\gamma_{\mathrm{SR},i}$ is the instantaneous SNR of the channel between the source and relay i. The average received SNR of the channels between S and D, and that between the relay i and D, $\bar{\gamma}_{\mathrm{SD}}$ and $\bar{\gamma}_{\mathrm{RD},i}$, can be obtained similarly.

B. Coded Cooperative DemF

This section briefly reviews coded cooperation as proposed for DemF based on punctured systematic repeat-accumulate (PSRA) codes [15], which is the basis for all further analysis in this paper. This channel code is used at both the source and relays because it only requires a simple encoder; in addition, these codes provide excellent performance in AWGN channels and the code rates can be changed easily. However, the method described below is valid for almost all systematic code families.

Encoding: The encoding procedure for PSRA codes can be found

in [15], [29], and is briefly reiterated here. Given an information bit string $\mathbf{w}$, which could consist of source bits (at the source) or demodulated bits (at the relay), we have the following procedure:

1) Repeat the information bits q times and randomly interleave;
2) Feed the permuted string into a truncated rate-1 recursive convolutional encoder with transfer function $1/(1+D)$ (i.e., accumulate, mod 2);
3) Concatenate the accumulated string with the original information sequence, so as to make the code systematic; and
4) Randomly puncture the *non-systematic* part of the codeword so as to achieve the desired code rate.

Throughout this paper, we set $q = 3$, hence allowing the code rate to range from 1/4 to 1.

Relay processing: As mentioned above, the ith relay forms a new codeword $\mathbf{d}_{R,i}$ based on the signal it observes, given by $\mathbf{y}_{\mathrm{SR},i}$. We admit two possible methods for forming $\mathbf{d}_{R,i}$:

1) For hardware that only admits operations with very low complexity, the observed sequence $\mathbf{y}_{\mathrm{SR},i}$ is merely demodulated. Since the modulation was BPSK, this is done by obtaining $\xi^{-1}(\mathrm{sign}(\mathbf{y}_{\mathrm{SR},i}))$. This sequence is then re-encoded with another error-correcting code. Given the above encoding procedure for PSRA codes, coded DemF involves only simple operations at the relay, and no decoding is involved. This would be appropriate for relay networks comprising nodes with only simple hardware and limited battery power.
2) For hardware that admits operations with higher complexity, the relay can perform *limited decoding*, for example, via limited iterations of SPA [25], or the Gallager A algorithm [26]. The partially decoded sequence, which may still contain errors, is re-encoded with another error-correcting code and retransmitted. The decoding operations at the relay are as specified in [25], [26].

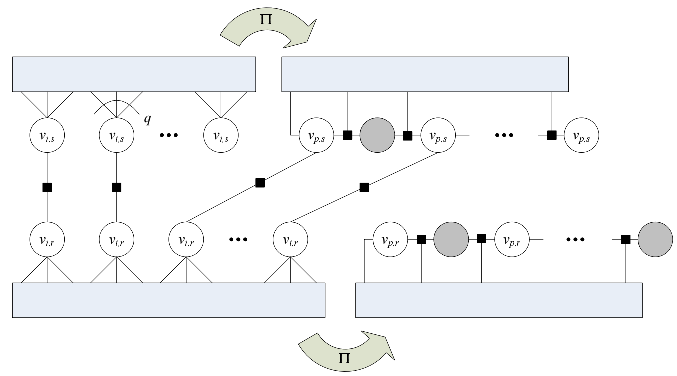

Fig. 1. Factor graph of DemF with the use of RA code.

In either case, we admit the possibility that *only a fraction* of the demodulated or decoded sequence is retransmitted; that is, the demodulated or partially decoded sequence is punctured prior to re-encoding; this is discussed further in the next section. Clearly, limited decoding is a generalized technique that admits pure DemF and pure DF as extreme points.

If limited decoding is available, the effective S-R channel is improved, as compared to the case of pure DemF. The relationship between the original S-R channel bit error rate (BER) and the *effective* S-R channel BER after decoding can be obtained using DE. As shown in [17], [24], the BER is a non-increasing function of the number of iterations. Hence, limited decoding does not increase the *effective* bit-flip probability over the S-R channel. As we will see, the BP can account for the impact of this limited decoding (via the effective S-R channel).

A factor graph [25] of DemF with use of a RA code is shown in Fig. 1. In the figure, circles and squares represent variable and check nodes respectively, where shaded variable nodes represent punctured parity bits. The labels $v_{i,s}$, $v_{p,s}$, $v_{i,r}$ and $v_{p,r}$ represent source information and parity bits, and relay information and parity bits respectively, and Π is the random interleaver. The relay parity bits only influence the decisions on the value of the source codeword through the relay information bits, and this is illustrated by the fact that none of the $v_{p,r}$ nodes are connected to any of the $v_{i,s}$ and $v_{p,s}$ nodes in the figure.

Decoding at the destination. We assume that the destination has knowledge of the channel statistics at each node. To simplify the analysis, we implement *serial decoding*, in which the destination first decodes the PSRA codes generated by each relay, and uses the results to decode the codeword transmitted by the source.

From (3), the destination observes each relay codeword $\mathbf{c}_{R,i}$ through independent Gaussian noise. Assuming serial decoding, each of these codewords is decoded individually. The decoding at the destination is performed using SPA [25], chosen because it is the usual algorithm for decoding "turbo-like" codes, such as the PSRA code. Furthermore, using the SPA, it is straightforward to explicitly account for the relationship between symbols transmitted by the source, and symbols transmitted by the relay.

For each relayed symbol, the sequence is equivalent to a symbol from $\mathbf{c}_S$, observed through a binary symmetric channel. For the ith relay, let p_i represent the crossover probability of the equivalent relay channel. The value of p_i can be found as the SNR between S and all the relays are known at D. This error probability is explicitly accounted for by calculating a special message in the SPA. After the final iteration in the process of decoding the relay codeword, the messages from all the check nodes to each variable node are summed. If the jth bit of $\mathbf{d}_S$ is relayed by relay i, let $l_{i,j}$ represent the summed message at the variable node in $\mathbf{d}_{R,i}$ corresponding to jth bit of $\mathbf{d}_S$; if the jth bit of $\mathbf{d}_S$ is not relayed by relay i, then $l_{i,j}$ is set to 0. After obtaining $l_{i,j}$ from decoding the codeword from relay i, the SPA message of the jth source bit from corresponding bit transmitted by the ith relay can be found using

$$\log \frac{p_i + (1-p_i)e^{l_{i,j}}}{p_i e^{l_{i,j}} + (1-p_i)}.$$

If $l_{i,j} = 0$, which means that bit j is not relayed by relay i, then the term vanishes. Also, if the S-R channel is unreliable, i.e., $p_i \to 0.5$, the above term approaches 0. This allows us to take into account the reliability of the S-R channel, and the effects of error propagation that are often seen with DF is eliminated. The channel message for bit j of $\mathbf{d}_S$, which is used for decoding, is then given by

$$\phi_j = \frac{4|h_{\mathrm{SD}}|y_{\mathrm{SD},j}}{N_{0,D}} + \sum_{i=1}^{r} \log \frac{p_i + (1-p_i)e^{l_{i,j}}}{p_i e^{l_{i,j}} + (1-p_i)}, \qquad (5)$$

where $y_{\mathrm{SD},j}$ is the jth bit of $\mathbf{y}_{\mathrm{SD}}$. Note that (5) represents channel observations of the same symbol through parallel independent channels, where the first term represents the direct channel, and the sum over all relays represents the equivalent relay channels.

The decoding process described above requires the knowledge of the S-R channel SNR (equivalently p_i) for all the relaying nodes. This can be facilitated by the relay estimating the S-R channel SNR, and transmitting the value to the destination node together with the relayed data. Performance degradation can be introduced by inaccurate channel estimation, and on D not recovering the estimated value from the relay. However, these effects are outside the scope of this paper. The decoding procedure at the destination node is discussed extensively in [15], and the reader is directed to that reference for additional details.

C. Fractional Cooperation

Most existing cooperation schemes assume "all or nothing" cooperation, i.e., either the relay devotes all its resources to the source or none at all. As we suggested in the previous section, a relay may use only a *fraction of its data block* to help a source. In a variation of this theme, multiple relays are recruited to assist the source node. Each relay transmits a random fraction of the complete codeword. The responsibility of relaying is then spread over multiple relays, thus reducing the risk of draining the battery of some of the nodes more quickly than others.

We introduce a new parameter, ϵ_i, to represent the fraction of the source codeword that relay i is relaying. As shown in [15], as long as the number of relays used, r, is greater than a required threshold r_c, then each additional relay increases the diversity order by 1. The value of r_c depends on the value of ϵ_i, the channel code used and the associated decoding scheme, as well as the method used by the relay to process the received source signal. Note that fractional cooperation is extremely flexible, as each relay chooses the source bits to relay at random and no coordination between the relaying nodes is required. We introduce some additional parameters to help describe the transmission scheme: in the case where a distinction is made between the relayed source information and parity bits, $\epsilon_{info,i}$ and $\epsilon_{par,i}$ describe the fraction of source *information* and *parity* bits that are relayed by relay i.

D. Error Rates and the BP

The BP approach provides a convenient upper bound on the maximum likelihood (ML) FER [30]. For a codebook $\mathcal{C}$, suppose the codewords are numbered $0, 1, 2, \ldots, |\mathcal{C}|-1$, and let $\mathbf{d}_i \in \mathcal{C}$ represent the ith codeword. Assume that codeword $\mathbf{d}_0 = \mathbf{0}$ is transmitted over the channel, and $\mathbf{y}$ is the received signal. Let $f(\mathbf{y}|\mathbf{d})$ be the likelihood function of $\mathbf{y}$ given $\mathbf{d}$ is transmitted. Then $\mathbf{Y}_j = \{\mathbf{y} : f(\mathbf{y}|\mathbf{d}_0) \le f(\mathbf{y}|\mathbf{d}_j)\}$ represents the set of received signals that would lead us to decode $\mathbf{d}_j$ as the correct codeword under ML decoding given $\mathbf{d}_0$ is sent. A frame error occurs whenever the received signal belongs to any of the set $\mathbf{Y}_j$ for $j = 1, \ldots |\mathcal{C}| - 1$, and the FER ($P_f$) is upper-bounded

$$P_f \le \sum_{j=1}^{|\mathcal{C}|-1} \Pr(\mathbf{y} \in \mathbf{Y}_j).$$

Recall that $\mathbf{y}$ is the received signal given that $\mathbf{d}_0$ is transmitted. Hence it can be shown

$$\Pr(\mathbf{y} \in \mathbf{Y}_j) = \sum_{\mathbf{y} \in \mathbf{Y}_j} f(\mathbf{y}|\mathbf{d}_0) \le \sum_{\mathbf{y} \in \mathbf{Y}_j} \sqrt{f(\mathbf{y}|\mathbf{d}_0) f(\mathbf{y}|\mathbf{d}_j)}.$$

Let $\mathbf{d}_0$ and $\mathbf{d}_j$ differ in h_j positions. Assuming discrete memoryless channels, after some manipulation we have [18]

$$\Pr(\mathbf{y} \in \mathbf{Y}_j) \le \prod_{i=1}^{h_j} \sum_{y \in \mathcal{Y}} \sqrt{f(y|0) f(y|1)}. \qquad (6)$$

where $\mathcal{Y}$ is the alphabet of the output, and $p(y|0)$ and $p(y|1)$ are the probability of y given 0 and 1 is sent respectively. The

BP associated with a channel is defined as

$$\beta \triangleq \sum_{y \in \mathcal{Y}} \sqrt{f(y|0)f(y|1)}. \tag{7}$$

For the binary symmetric channel (BSC) with bit-flip probability p, $\beta = 2\sqrt{p(1-p)}$, and for the AWGN channel with received SNR γ, $\beta = e^{-\gamma}$. The BP therefore *characterizes the channel* between the transmitted symbol and the data used for processing, y.

Let k be the length of the of the encoder input, and n be the encoder output. By averaging over all possible codebooks, we obtain the union bounds for the P_f using the BP

$$P_f \leq \sum_{h=1}^{n} A_h \beta^h = \sum_{h=1}^{n} \left(\sum_{w=1}^{k} A_{w,h} \right) \beta^h \tag{8}$$

where $A_{w,h}$ represents the number of codewords with input weight w and output weight h, and A_h, also known as the weight enumerator (WE), is the number of codewords with weight h. If the channel code is systematic, $A_{w,h}$ represents the number of codewords with w as the weight of the information bits, and h as the weight of the codeword. The derivation of WE for "turbo-like" codes, such as RA codes, can be found in [14].

Importantly, for our application with relay channels, these bounds on the FER can be extended to scenarios where the codeword is sent through parallel channels, each with different BP [31]. Let the number of parallel channels be J. Assuming the codeword has n bits, and each bit is transmitted through channel j with probability α_j, where $\sum_j \alpha_j = 1$. By averaging over all possible bit assignment to all the channels, the authors in [31] derived the union bound for the parallel channels

$$P_f \leq \sum_{h=1}^{n} A_h \bar{\beta}^h \tag{9}$$

where

$$\bar{\beta} = \sum_{j=1}^{J} \alpha_j \beta_j \tag{10}$$

and β_j is the BP associated with the jth channel. This concept has also been extended to analyze incremental redundancy (IR) in [32], a cooperative scheme used in relay channels. With IR, the source codeword is divided into mutually-exclusive subsets, where each subset is assigned to one relay. After receiving the broadcast from S, each relay attempts to decode the source codeword. If decoding is successful, R repeats the subset of bits assigned; otherwise, R remains silent and S transmits the subset of bits assigned to that particular relay. This approach is different from that suggested in this paper, as the system under study can use DemF, and we do not assume that the subsets are mutually-exclusive. Note that this section only provides a brief overview of the union bound and BP. A more detailed description of the derivation of the union bound can be found in [18], [30], [33], [34].

III. Applications of the BP in Relay Channels

This section presents the core contributions of this paper: the application of the BP to relay selection and fractional cooperation in the context of DemF. These topics were the background to our investigation of the BP. However, it is our belief that the approach presented here can be used in several other analysis and design scenarios. Because we use a Gaussian distribution to approximate the density of the SPA messages passed when decoding the PSRA code, our bound is itself an approximation. Nonetheless, analogously to the density approximation for LDPC codes, the bound can be calculated quickly and efficiently, making it most appropriate for simplified performance analysis and system design.

A. Selecting a Single Relay

Error performance is used as the figure of merit in this paper, and we wish to select the relay that results in the smallest FER. For simplicity, the subscript i that is used to identify the different relays is omitted in this section as only one relay is chosen to assist the source. Without loss of generality, the cooperative RA code in Sec. II-B is used to explain the use of BP in relay selection. Let k_S be the number of information bits in the source codeword, and recall that n_S is the blocklength of the source codeword. In addition we assume that a rate-1/4 systematic RA code is used by the relay and that serial decoding is performed.

Recall that for a systematic code, $A_{w,h}$ is the number of codewords with weight w for the information bits, and weight $h-w$ for the parity bits. In the application of (9) to the 3-node relay channel, we have

$$P_f \leq \sum_{w=1}^{k} \sum_{h=1}^{n} A_{w,h} \beta_{\mathrm{SD}}^h \times (1-\epsilon_{info}(1-\beta_R))^w (1-\epsilon_{par}(1-\beta_R))^{h-w} \tag{11}$$

where β_{SD} is the BP associated with the S-D channel and β_R is the BP associated with the relay used for forwarding the source signal. From the source codeword viewpoint, it is first transmitted through the S-D channel, with the associated BP $\beta_{\mathrm{SD}} = e^{-\gamma_{\mathrm{SD}}}$. Then, for the bits that are relayed, they are transmitted through the compound S-R-D channel, with the associated BP β_R.

As shown in (11), for a fixed code and a fixed relaying scheme, $A_{w,h}$, ϵ_{info} and ϵ_{par} remain constant, and the bound on the FER decreases as the BP decreases. When choosing the optimal relay, the BP β_{SD} is fixed (as the S-D channel is the same for all relays); the only value that is distinct for different relays is β_R. Hence, as β_R decreases, the upper bounds on the FER go down. Although it cannot be shown that choosing the relay with a link that gives the minimum β_R will provide the smallest FER as well, we expect that using the chosen relay will provide us with good error performance.

As serial decoding is used at the destination node, the calculation of β_R can be separated into two steps. In the first step, we need to obtain the distribution of the output signal after decoding the *relay* codeword. Let the likelihood function for the R-D channel *before* decoding be modeled as

$$\begin{aligned} f(y|d_R=0) &= g(y;1,\sigma_{R,in}^2) \\ f(y|d_R=1) &= g(y;-1,\sigma_{R,in}^2) \end{aligned} \tag{12}$$

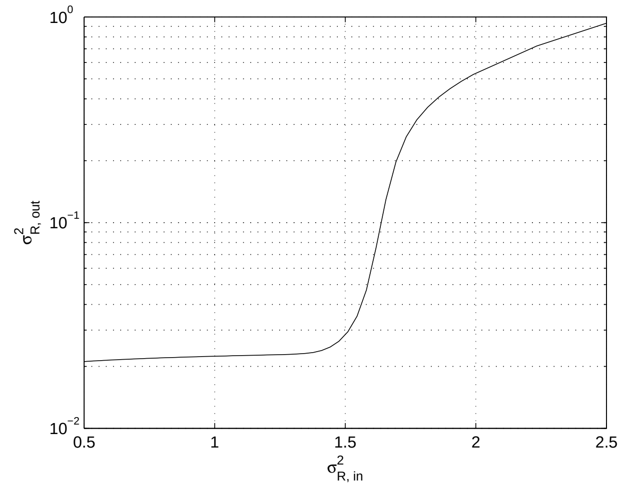

Fig. 2. Plot of equivalent channel noise variance before and after decoding for the rate-1/4 systematic RA code.

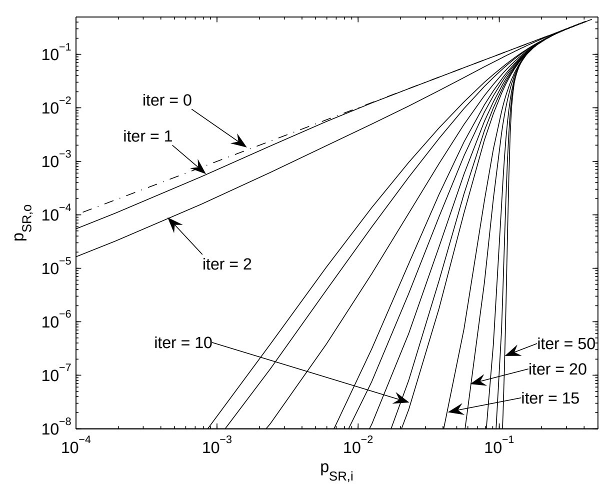

Fig. 3. Relationship between $p_{\mathrm{SR},in}$ and $p_{\mathrm{SR},out}$, the equivalent bit-flip probability after a given number of decoding iterations for the rate-1/2 punctured systematic RA code.

where $g(t;\mu,\sigma^2)$ is the distribution of the Gaussian random variable t with mean μ and variance σ^2. In our case, because no puncturing is used at the relay codeword, the distribution of the output signal after decoding can also be (fairly accurately) approximated by the Gaussian distribution with associated parameters found using the technique from [23]. Let the output distribution for the R-D channel *after* decoding be modeled as

$$\begin{aligned} f(y|d_R=0) &= g(y;1,\sigma^2_{R,out}) \\ f(y|d_R=1) &= g(y;-1,\sigma^2_{R,out}). \end{aligned} \tag{13}$$

The relationship between the variance of the channel distribution before and after decoding the rate-1/4 RA code is plotted in Fig. 2, where $\sigma^2_{R,in} = \frac{|h_{\mathrm{RD}}|^2}{N_{0,D}}$ is the variance for the distribution in (12) over the R-D channel before decoding, and $\sigma^2_{R,out}$ is the equivalent variance after decoding. This data is stored at the relay node, where the equivalent channel noise variance $\sigma^2_{R,out}$ can be found through a lookup table given the R-D channel coefficient. Note that this look up table has to be created just once for a particular code, i.e., DE in real time is not required. Furthermore, the required lookup table is one-dimensional, and the storage of such data can be easily implemented.

In the second step, we use the information obtained from the first step, $\sigma^2_{R,out}$, together with the S-R channel condition, to calculate β_R. Recall that the relay can process the received signal in one of two ways: either it merely demodulates the received signal, or it performs limited decoding on the received signal. If only demodulation is performed at the relay, the S-R channel can be modeled as a BSC with bit-flip probability $p_{\mathrm{SR},in}$, where $p_{\mathrm{SR},in} = \frac{1}{2}\mathrm{erfc}\sqrt{\gamma_{\mathrm{SR}}}$, and $\mathrm{erfc}(\cdot)$ is the complementary error function. If limited decoding is performed at the relay, then the bit flip probability after relay processing is not equal to $p_{\mathrm{SR},in}$; let $p_{\mathrm{SR},out}$ represent this new bit flip probability after decoding, which can be obtained using DE.

As an example, the relationship between the raw bit-flip probability $p_{\mathrm{SR},in}$ and the equivalent bit-flip probability $p_{\mathrm{SR},out}$ *after* a given number decoding iterations for a rate-1/2 punctured systematic RA code is illustrated in Fig. 3. In the figure, each curve represents a different number of decoding iterations; the results for 1 to 10, 15, 20, 30, 40 and 50 iterations are shown. As the number of iterations increases, $p_{\mathrm{SR},out}$ decreases for the same $p_{\mathrm{SR},in}$. The result for no iterations, where only demodulation is performed at the relay and $p_{\mathrm{SR},out} = p_{\mathrm{SR},in}$, is shown in the plot for comparison (represented by the *dash-dot* line). As illustrated in the plot, even a small number of iterations can improve the bit error rate over the S-R channel significantly. These values can be obtained at the relay either from a function approximating the relationship, or, similar to $\sigma^2_{R,out}$, from a lookup table.

After the equivalent channel noise variance $\sigma^2_{R,out}$ and equivalent bit-flip probability $p_{\mathrm{SR},out}$ are obtained, the likelihood function for the equivalent S-R-D channel after decoding the relay codeword is given by

$$f(y|d_S) = \begin{cases} g(y;1,\sigma^2_{R,out})\bar{p}_{\mathrm{SR},out} \\ \quad + g(y;-1,\sigma^2_{R,out})p_{\mathrm{SR},out} & \text{if } d_S=0, \\ g(y;1,\sigma^2_{R,out})p_{\mathrm{SR},out} \\ \quad + g(y;-1,\sigma^2_{R,out})\bar{p}_{\mathrm{SR},out} & \text{if } d_S=1. \end{cases} \tag{14}$$

where $\bar{p}_{\mathrm{SR},out} = 1 - p_{\mathrm{SR},out}$. The relay then calculates β_R by substituting (14) into (7)

$$\beta_R = \int_{-\infty}^{\infty} \sqrt{f(y|d_S=0)f(y|d_S=1)}dy \quad (15a)$$

$$= \int_{-\infty}^{\infty} \{[g(y;1,\sigma_{R,out}^2)\bar{p}_{\mathrm{SR},out} + g(y;-1,\sigma_{R,out}^2)p_{\mathrm{SR},out}] \times [g(y;-1,\sigma_{R,out}^2)\bar{p}_{\mathrm{SR},out} + g(y;1,\sigma_{R,out}^2)p_{\mathrm{SR},out}]\}^{1/2}dy \quad (15b)$$

$$= \frac{1}{\sqrt{2\pi\sigma_{R,out}^2}} \int_{-\infty}^{\infty} \exp\left\{-\frac{y^2+1}{2\sigma_{R,out}^2}\right\} \times [4p_{\mathrm{SR},out}\bar{p}_{\mathrm{SR},out}\sinh^2(y/\sigma_{R,out}^2)+1]^{1/2}dy. \quad (15c)$$

Note that as $p_{\mathrm{SR},out}$ becomes very small, and the S-R-D channel becomes an additive white Gaussian noise channel, with $\beta_R = \exp\{-\frac{1}{2\sigma_{R,out}^2}\}$. Similarly, as $\sigma_{R,out}^2$ becomes very small, the S-R-D channel becomes a BSC, with $\beta_R = 2\sqrt{p_{\mathrm{SR},out}\bar{p}_{\mathrm{SR},out}}$.

A comparison of the results from DE on the complete factor graph and from the lookup table with calculation from (15c) is shown in Fig. 4. The parameters ϵ_{info} and ϵ_{par} are set to be 1 and 0, and the rate of the source codeword is $R_S = 1/2$. It is assumed that only demodulation is performed at the relay. In addition, the S-D channel instantaneous SNR γ_{SD} is set to -6 dB. In the figure, the markers indicate the values of γ_{SR} and γ_{RD} that yield the given BER through the use of DE on the factor graph, and the lines show the contours outlining values of γ_{SR} and γ_{RD} that give the same β_R values found using the lookup table and (15c). As shown in the plot, the calculation of β_R through the lookup table and (15c) does an excellent job of characterizing the S-R and R-D channel conditions that would yield a certain BER. This is quite remarkable, as all the channel conditions that give the same β_R also give the same BER. The plot, therefore, illustrates that the BP can indeed be used for relay selection; it is shown that as β_R decreases, the value of BER goes down as well. The BP can therefore be used as a simple and efficient *performance measure*. We emphasize that the calculations described above place a very limited real-time computation burden on the relays.

Because of the use of orthogonal channels, the S-D channel is not required to perform relay selection. As only knowledge of the S-R and R-D channel condition is required, the BP calculation can be performed at each relay. For example, after S has sent out a request for assistance, each available relay calculates its associated BP, and sends this information to either S or D, depending on which node is responsible for choosing the relay. After collecting all the values of β_R from the relays, S or D then chooses the best relay. There is, however, the need for the relay to keep track of the R-D channel condition. We assume that pilot signals are transmitted from each relay to the destination node periodically to keep track of the R-D channel coefficient, and as the D is not energy limited, it can broadcast these channel coefficient estimates to the relays. The penalty that may be incurred due to inaccurate channel estimation, however, is outside the scope of this paper.

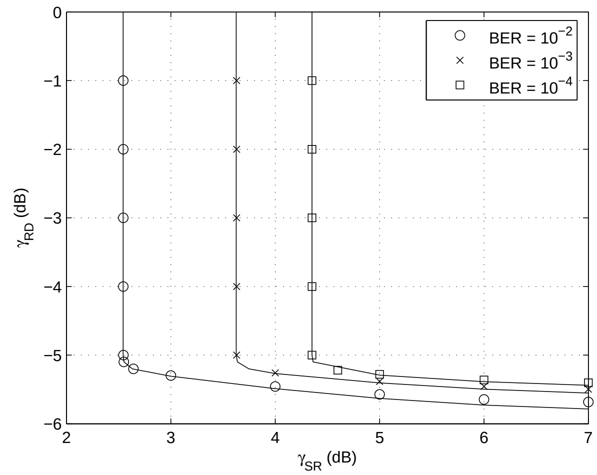

Fig. 4. Contours of BER from DE (*Markers*) and from lookup table and (15c) (*Lines*).

B. Relay Selection for Fractional Cooperation

In extending the BP approach to fractional cooperation we must first take a detour into some theoretical results when relay selection is applied to fractional cooperation. These results show that, as with DF and AF, relay selection provides "full" diversity order. As in the earlier section, BP can then be used to select the relays.

An analysis of fractional cooperation was presented in [15], and the results are briefly reviewed here. Let $r_c(\gamma_{\mathrm{SD}})$ represent the number of relays such that whenever the number of relays used, r, is less than $r_c(\gamma_{\mathrm{SD}})$, a system outage always occurs when the instantaneous S-D channel SNR is γ_{SD} and for all S-R and R-D channels. It was shown that $r_c(\gamma_{\mathrm{SD}})$ exists for all γ_{SD} and is finite. Let $r_c = \lim_{\gamma_{\mathrm{SD}}\to 0} r_c(\gamma_{\mathrm{SD}})$ be the minimum number of relays that must be participating in order for the system to have diversity order greater than 1. It was shown that if $r < r_c$, then the system has diversity 1. On the other hand, if $r \geq r_c$, then a diversity order of $r - r_c + 2$ can be achieved with the use of fractional cooperation. Essentially the theorem states that if the number of relays is above a threshold, each additional relay provides an extra order of diversity. The value of r_c is dependent on the value of ϵ_i, the coding and decoding schemes used, as well as the relaying scheme.

These results are extended here to the case of relay selection. We assume that r_b "best" relays are chosen out of a pool of r_a relays, where in this case the relays are chosen based on their ability to improve the system error performance. Each chosen relay then selects, at random, a fraction of the source code bits, form a new codeword based on the chosen bits, and transmit the codeword to the destination.

Now we provide some definitions for the terms that will be used. In a wireless system, an *outage* is declared when the FER falls above a given threshold. Let $r_t(\gamma_{\mathrm{SD}})$ be the number of available relays, such that if $r_a < r_t(\gamma_{\mathrm{SD}})$ then a system outage will occur, for that value of γ_{SD} and all S-R and R-D channels, independently of the value of r_b. When $r_a \geq r_t(\gamma_{\mathrm{SD}})$ relays are available to assist, but the number of relays chosen, r_b, is less than a threshold $\tilde{r}_c(\gamma_{\text{SD}}, r_a)$, then an outage would occur for those values of γ_{SD} and r_a. In addition, let $r_t = \lim_{\gamma_{\text{SD}}\to 0} r_t(\gamma_{\text{SD}})$ be the minimum r_t without a S-D channel, and $\tilde{r}_c(r_a) = \lim_{\gamma_{\text{SD}}\to 0} \tilde{r}_c(\gamma_{\text{SD}}, r_a)$.

Lemma 1: For all γ_{SD}, $r_t(\gamma_{\text{SD}})$ exists and is finite. Also, for all γ_{SD} and $r_a > r_t(\gamma_{\text{SD}})$, $\tilde{r}_c(\gamma_{\text{SD}}, r_a)$ exists and is less than or equal to r_a.

Proof: See Appendix A. ∎

To study the asymptotic outage probability, P_{out}, the order notation $\Theta(\cdot)$ is used here. If $y = \Theta(x)$, $\lim_{x\to\infty} y/x = c$, where c is a constant. For example, for a system with diversity order d, $P_{out} = \Theta(\bar{\gamma}^{-d})$.

Theorem 1: If $r_a < r_t$, the diversity order of the system is 1. For $r_a \geq r_t$, if $r_b < \tilde{r}_c(r_a)$, then the diversity order is 1 as well. If $r_a \geq r_t$ and $r_b \geq \tilde{r}_c(r_a)$, then the diversity order of the system is $r_a - \tilde{r}_c(r_a) + 2$.

Proof: See Appendix B. ∎

Similar to the one-relay system, relay selection can be used to choose the best r_b relays out of a pool of r_a available relays to provide full diversity. Using the same technique as presented earlier, β_R values of each relay can be calculated, and the r_b relays with the smallest β_R values is chosen to assist the source. If $r_a \geq r_t$ and $r_b \geq \tilde{r}_c(r_a)$, then a diversity order of $r_a - \tilde{r}_c(r_a) + 2$ can be observed. Note that only $\tilde{r}_c(r_a)$ relays is required to obtain that maximum diversity order for a pool of r_a available relays; any additional relay chosen to assist only shifts the FER curve and does not provide an increase in diversity order.

C. Outage Probability Analysis for Fractional Cooperation

Sections III-A and III-B illustrated a system design application of the BP. In addition to relay selection, the BP can also be used for analysis; here, to analyze the outage probability of fractional cooperation. Simulations and DE can be used to obtain the outage probability, but they are complex and time-consuming. As illustrated earlier, the union bound provides an upper bound on the FER, and is more efficient than simulations or DE. Assume that r relays are used, and relay i chooses to relay each of the n bits of the source with probability ϵ_i. In this case, there is no differentiation between the fraction of information and parity bits relayed, and $\epsilon_{info,i} = \epsilon_{par,i} = \epsilon_i$.

Let $\mathfrak{R} = \{1, 2, \ldots, r\}$ be the set of relays, and let $\mathcal{R}_j$ denote all possible subsets of $\mathfrak{R}$, including the empty set $\emptyset$, for $j = 1, \ldots, 2^r$. Then the probability of each bit of the source codeword relayed only by the relays in set $\mathcal{R}_j$ is given by

$$\Pr(\mathcal{R}_j) = \left(\prod_{i\in\mathcal{R}_j} \epsilon_i\right)\left(\prod_{i\notin\mathcal{R}_j} (1-\epsilon_i)\right) \tag{16}$$

and the associated BP is given by

$$\beta_{\mathcal{R}_j} = \prod_{i\in\mathcal{R}_j} \beta_{R,i} \tag{17}$$

where $\beta_{R,i}$ is the BP associated with the S-R-D link through relay i. After some simple manipulation, it can be shown that $\bar{\beta}$ from (10) for fractional cooperation is given by

$$\bar{\beta}_{FC} = \prod_{i=1}^{r} \left(1 - \epsilon_i \left(1 - \beta_{R,i}\right)\right) \tag{18}$$

If outage is declared when the channel gives $P_f > P_{f,t}$, where $P_{f,t}$ is a predefined FER threshold, then

$$\begin{aligned} P_{out} &= \Pr(P_f > P_{f,t}) \\ &\leq \Pr\left(\sum_{h=1}^{n} A_h \left(\beta_{\text{SD}} \prod_{i=1}^{r} \left(1 - \epsilon_i\left(1-\beta_{R,i}\right)\right)\right)^h > P_{f,t}\right) &\text{(19a)} \\ &= \Pr\left(\beta_{\text{SD}}\bar{\beta}_{FC} > \beta_t\right) &\text{(19b)} \end{aligned}$$

where $P_{f,t} = \sum A_h \beta_t^h$.

To obtain the upper bound on P_{out}, random realizations of the S-D, S-R and R-D channels are first generated according to their distributions. The values of β_{SD} and the $\beta_{R,i}$ associated with the relays are found. The frequency at which $\beta_{\text{SD}}\bar{\beta}_{FC}$ exceeds β_t gives the upper bound on the outage probability. If simulations were used to obtain the outage probability, the FER of each realization of the channels must be found, and this is far more time-consuming and complex than the calculations in (18). DE is more efficient than simulations, but is relatively time-consuming and complex as well. Hence, using (19b) is by far one of the most efficient methods to approximate the performance of a relay channel using fractional cooperation, indeed any cooperative scheme.

D. Limitations

This study of design and analysis is done here under two major assumptions: the relay codeword is not punctured, and serial decoding is performed at the destination. The first assumption simplifies the required analysis. When the codeword is not punctured, the log-likelihood ratio output from decoding the relay codeword can be approximated closely by the Gaussian distribution, allowing us to model the equivalent channel distribution after decoding with a Gaussian distribution. If puncturing were used on the relay codeword, the equivalent channel distribution after decoding resembles a mixture of Gaussians. The distribution can still be approximated with a Gaussian distribution [23], but the approximate description is not as accurate as in the case without puncturing, and might lead to performance degradation in the case of relay selection, and possibly inaccurate analysis. The effect of this approximation, however, is outside the scope of this paper.

Similarly, the second assumption that serial decoding is performed at the destination simplifies the analysis. Under this assumption, the BP can then be computed in two steps: first, find the effect of decoding the relay codeword, and second find the effect of the imperfect S-R channel. As developed here, BP cannot be used for relay selection or analysis when parallel decoding is used at the destination node, where the source and relay codewords are decoded simultaneously with the SPA messages are passed between the two encoders after each iteration. The use of BP under parallel decoding is more complex, and is left to future work.

Finally, the union bound closely follows the error performance only at high SNR, and otherwise it is a fairly loose bound. Tighter bounds can be used to provide a better approximation of the performance of various relaying schemes, and examples of these bounds can be found in [31], [35].

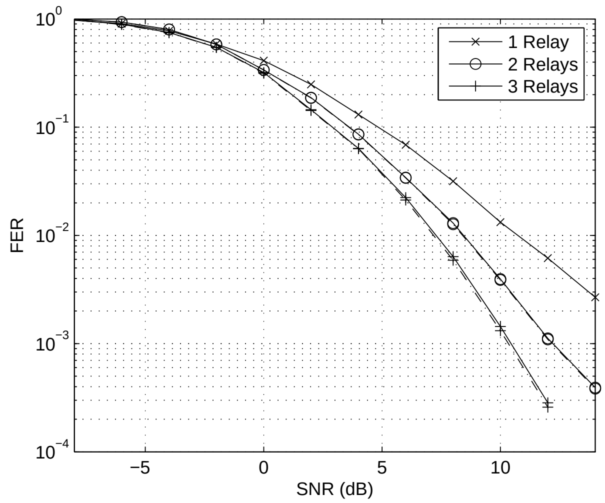

Fig. 5. FER for our relay selection scheme (*solid lines*) and exhaustive search (*dash-dot lines*).

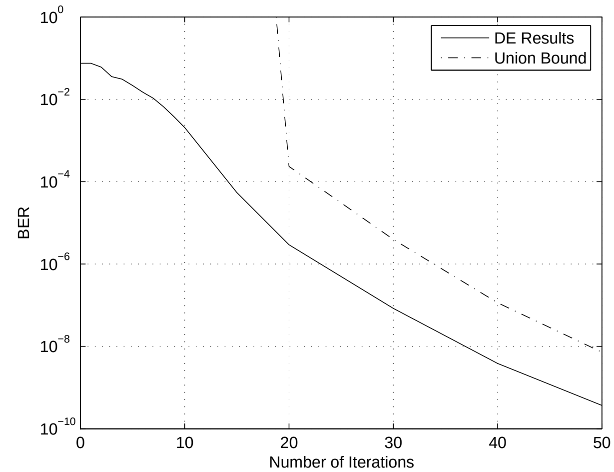

Fig. 6. BER and upper bound on BER are shown for various number of SPA iterations performed at the relay with $\gamma_{\rm SD} = -6$ dB, $\gamma_{\rm SR} = -1$ dB and $\gamma_{\rm RD} = -4$ dB.

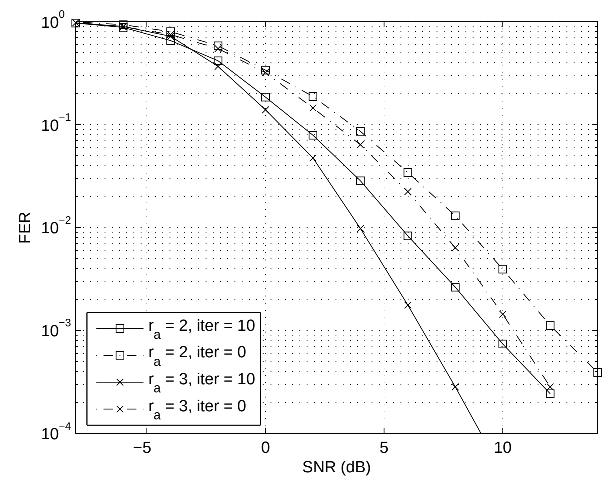

Fig. 7. FER for our relay selection scheme where 10 iterations of SPA are performed in the decoders at the relays.

These better approximations, however, come at a price of more complex calculations. However, despite these assumptions and limitations, the BP approach is still an efficient method that can be used to observe the effect of various parameters without resorting to simulations or DE, which are comparatively time-consuming and complex.

IV. SIMULATIONS

This section presents the results of simulations to illustrate the efficacy of the BP approach. The simulations cover the applications mentioned here: relay selection, limited decoding, and outage in fractional cooperation.

First we show that relay selection using the BP approach provides performance very close to optimal. In this example, we set $\epsilon_{info} = 1$ and $\epsilon_{par} = 0$, and pure demodulation is performed at the chosen relay. The rate of the source code word is $R_S = 1/2$, with $n_S = 4000$, and the rate of the relay codeword is $R_R = 1/4$, with blocklength $n_R = 8000$. All the channels have the same average received SNR. The FER for 1 relay and choosing 1 relay out of 2 or 3 available relays using the lookup table and (15c) is shown in Fig. 5. The figure also shows simulation results for the case where for a given S-D channel, 2 or 3 relays are available to assist in the transmission, and the relay with the least number of bit errors is used. These results (by exhaustive search) are used as the optimal case, and provides a lower bound on the FER performance. Note that, as expected, selection cooperation provides full diversity order despite the fact that only one relay is used. As illustrated in the plot, the FER for exhaustive search and our low complexity relay selection scheme are essentially the same, showing the relay selection scheme using BP provides excellent performance in minimizing the FER.

The performance improvement with limited decoding at the relay is shown in Fig. 6. The SNR of the AWGN channels are $\gamma_{\rm SD} = -6$ dB, $\gamma_{\rm SR} = -1$ dB and $\gamma_{\rm RD} = -4$ dB. In the plot, BER obtained using DE and upper bound on the BER found using the BP approach are illustrated. It is shown that by performing small number of decoding iterations, e.g., by allowing the relay to perform 10 iterations of the SPA before relaying the source bits, the BER can be reduced by more than a factor of 100. Also, the union bound follows the BER curve closely for small BER. The FER improvement with the use of limited decoding is illustrated in Fig. 7. The settings for the fading channel simulation is same as for that from Fig. 5, but here it is assumed that all the relays can perform 10 iterations of SPA before forming a hard decision on the relayed bits. The FER curves from Fig. 5 are also shown here for comparison, and it can be seen that only 10 iterations of SPA can provide a gain of as much as 3 dB. For the simulation results shown in Fig. 6 and 7, the effective $p_{SR,out}$ after decoding is found using DE.

In Fig. 8, simulation results for fractional cooperation with relay selection are shown. All the channels have the same average SNR. Here we set $\epsilon_i = 0.2$, and $r_a = 8$. Again the rate of the source code word is $R_S = 1/2$, with $n_S = 4000$,

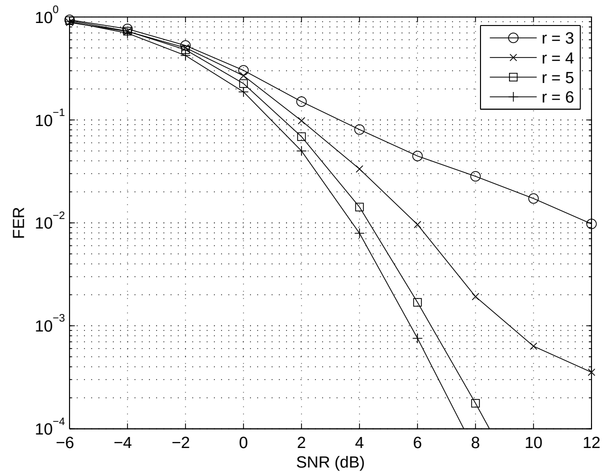

Fig. 8. FER of fractional cooperation with relay selection, where $r_a = 8$ and $\epsilon_i = 0.2$.

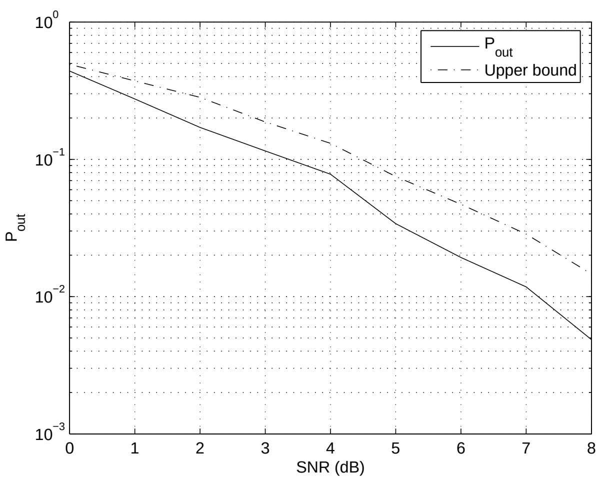

Fig. 9. Outage probability from simulation and upper bound on outage probability, where $r = 4$, $\epsilon_i = 0.3$ and $P_{f,t} = 0.01$.

and the rate of the relay codeword is $R_R = 1/4$. From the plot, it can be seen that if $r_b < 5$, the diversity order at high SNR is 1, and when $r_b \geq 5$, the diversity order is 5. Recall that in Theorem 1, we have shown that if $r_b < \tilde{r}_c(r_a)$, then the diversity order is 1 at high SNR, whereas if $r_b \geq \tilde{r}_c(r_a)$, the diversity order is $r_a - \tilde{r}_c(r_a) + 2$. Hence it can be deduced from the plot that for $\epsilon = 0.2$ and $r_a = 8$, $\tilde{r}_c(r_a) = 5$.

The use of the BP to obtain an upper bound on the outage probability is shown in Fig. 9. Here $R_S = 1/2$, $n_S = 1000$ and $R_R = 1/4$. Similar to the previous plot, all the channels have the same average SNR. It is also assumed that $r = 4$, $\epsilon_i = 0.3$ and $P_{f,t} = 0.01$. Both simulation results and the upper bound obtained using (19b) are shown. In this case, the upper bound can be used to approximate the performance of fractional cooperation in fading channels. Note that obtaining this upper bound via the BP is very efficient when compared to obtaining the results through simulation.

V. Conclusions

As illustrated in the simulations above, the BP presents several advantages in terms of efficient system design and analysis in relay channels where DemF relaying scheme is used. First, assuming that the FER is used as a system measure, it can be used for relay selection. We have shown that the BP approach can be used in both scenarios where either one relay is chosen, or multiple relays are chosen while fractional cooperation is used. In addition to relay selection, the union bound, together with the BP, can be used to provide an efficient method to obtain a close approximation to the upper bound on the outage probability. This is done by generating random realizations of the various fading channels, and observing the frequency at which the upper bound on the FER exceeds the threshold FER. This is more efficient than obtaining the FER from either simulations or DE as explained earlier. With the use of the BP approach, the effects of decoding at the relay can be taken into account in relay selection and analysis. In summary, this paper has presented the BP as a proxy for system error performance when DemF is used, allowing for efficient design and analysis of distributed wireless networks. Here we have applied this approach to problems that arose in our previous work. However, we believe that this BP-based approach has applications beyond those considered here.

Appendix A
Proof of Lemma 1

Proof: This proof closely follows the steps used to prove Lemma 1 in [15]. For simplicity, we assume that the fraction of the source codeword relayed are the same for all relays, where $\epsilon_i = \epsilon$. We will also assume that $\gamma_{\mathrm{SD}} = 0$. In addition, if $\gamma_{\mathrm{SR},i}$ or $\gamma_{\mathrm{RD},i}$ falls below the threshold $\gamma_{\min}$ then relay i is not used for relaying. Let $p_a = \Pr(\gamma_{\mathrm{SR},i} > \gamma_{\min} \cap \gamma_{\mathrm{SR},i} > \gamma_{\min})$. Given that the r_b best relays are chosen out of a pool of r_a relays, then the probability of a bit not chosen by *any* of these relays is

$$\begin{aligned} p_{nr} &= \sum_{k=0}^{r_b} \binom{r_a}{k} (1-p_a)^{r_a-k} p_a^k (1-\epsilon)^k \\ &\quad + (1-\epsilon)^{r_b} \sum_{k=r_b+1}^{r_a} \binom{r_a}{k} (1-p_a)^{r_a-k} p_a^k \qquad (20a) \\ &= \sum_{k=0}^{r_a} \binom{r_a}{k} (1-p_a)^{r_a-k} p_a^k (1-\epsilon)^k \\ &\quad + \sum_{k=r_b+1}^{r_a} \binom{r_a}{k} (1-p_a)^{r_a-k} p_a^k \left[(1-\epsilon)^{r_b} - (1-\epsilon)^k\right] \qquad (20b) \\ &= [1-\epsilon p_a]^{r_a} \\ &\quad + \sum_{k=r_b+1}^{r_a} \binom{r_a}{k} (1-p_a)^{r_a-k} p_a^k \left[(1-\epsilon)^{r_b} - (1-\epsilon)^k\right]. \qquad (20c) \end{aligned}$$

Since the second term in (20c) is always positive, given a fixed r_a, p_{nr} decreases as r_b increases, and the smallest p_{nr} is reached when $r_b = r_a$.

Using similar arguments as in [15], the FER is upper bounded by

$$1 - (1 - p_{nr} - 2(1 - p_{nr})\zeta\,(1 - \zeta))^n\,, \tag{21}$$

where

$$\zeta = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\gamma_{\min}}\right),$$

and where n is the length of the source codeword. Let $r_b = r_a$, and the second term in (20c) vanishes. There exists $\gamma_{\min}$ large enough so that $\zeta \to 0$. Increasing $\gamma_{\min}$ reduces p_a, but r_a can be increased such that $p_{nr} \to 0$. So any frame error rate criterion for outage probability can be satisfied, for some value of r_a (denoted as r_t). Now assume that $r_a \geq r_t$, then there exists a $r_b \leq r_a$ large enough such that the FER criterion can be satisfied, where the value of r_b satisfying the criterion is denoted as $\tilde{r}_c(r_a)$.

Now we will show that the result holds if we relax the assumptions stated at the beginning of the proof. As illustrated in the proof in [15], it can be shown that for the cases where $\gamma_{SD} > 0$, the results stated above holds. Now consider the case where the fraction relayed by each relay are not necessarily the same. First, we let $\epsilon_{\min} = \min_i \epsilon_i$. Consider two different systems, where in System A, relay i relays fraction ϵ_i of the source codeword, and in System B, all relays relay $\epsilon_{\min}$ of the source codeword. For System B, it has been shown that a corresponding $r_t(\gamma_{\mathrm{SD}})$ and $\tilde{r}_c(\gamma_{\mathrm{SD}}, r_a)$ can be found. Note that the p_{nr} associated with System A is smaller by the p_{nr} associated with System B for the same values of r_a and r_b. Hence the $r_t(\gamma_{\mathrm{SD}})$ and $\tilde{r}_c(\gamma_{\mathrm{SD}}, r_a)$ associated with System A are upper-bounded by those associated with System B, which are finite. This shows that the corresponding $r_t(\gamma_{\mathrm{SD}})$ and $\tilde{r}_c(\gamma_{\mathrm{SD}}, r_a)$ also exist when the fraction relayed by each relay are not necessarily identical. ■

APPENDIX B
PROOF OF THEOREM 1

Proof: This proof, again, follows closely the steps that were used to prove Theorem 1 in [15]. For simplicity, we have set the average SNR over all channels to $\bar{\gamma}$.

From the definition of r_t, if $r_a < r_t$, then the system is in outage if the S-D link is not present. Hence with the S-D link, the system has diversity order of 1. Also, from the definition of $\tilde{r}_c(r_a)$, if $r_a \geq r_t$, but $r_b < \tilde{r}_c(r_a)$, then the system has diversity order of 1 as well. For $r_a \geq r_t$ and $r_b \geq \tilde{r}_c(r_a)$, we will show that a diversity order of $r_a - \tilde{r}_c(r_a) + 2$ can be achieved. Let $\gamma_{\mathrm{SD},0} = \max\{\gamma_{\mathrm{SD}} : \tilde{r}_c(\gamma_{\mathrm{SD}}, r_a) = \tilde{r}_c(r_a)\}$ be the SNR required on the SD channel such that for any $\gamma_{\mathrm{SD}} < \gamma_{\mathrm{SD},0}$, $\tilde{r}_c(\gamma_{\mathrm{SD}}, r_a) = \tilde{r}_c(r_a)$. For each number of relays $r_b \geq \tilde{r}_c(r_a)$, there exists $\gamma_{\mathrm{suf},r} > 0$ such that if $\gamma_{\mathrm{SD}} < \gamma_{\mathrm{SD},0}$ and at least $r_b - \tilde{r}_c(r_a) + 1$ of the r_b chosen relays have $\gamma_{\mathrm{SR},i} < \gamma_{\mathrm{suf},r}$ or $\gamma_{\mathrm{RD},i} < \gamma_{\mathrm{suf},r}$, then an outage occurs. Note that if at least one of the r_b chosen relays are in outage, i.e., at least one of the r_b chosen relays have $\gamma_{\mathrm{SR},i} < \gamma_{\mathrm{suf},r}$ or $\gamma_{\mathrm{RD},i} < \gamma_{\mathrm{suf},r}$, that implies that *all* of the $r_a - r_b$ relays that are not chosen must be in outage as well, since the chosen r_b relays are the best out of the r_a available relays. Let

$$p_{o,r} := \Pr(\gamma_{\mathrm{SR},i} < \gamma_{\mathrm{suf},r} \cup \gamma_{\mathrm{RD},i} < \gamma_{\mathrm{suf},r}). \tag{22}$$

As shown in [15], $\Pr(\gamma_{\mathrm{SD}} < \gamma_{\mathrm{SD},0}) = \Theta(\bar{\gamma}^{-1})$, $p_{o,r} = \Theta(\bar{\gamma}^{-1})$ and $1 - p_{o,r} = \Theta(1)$. A lower bound on the probability of outage P_{out} is

$$\begin{aligned} P_{\mathrm{out}} &\geq \Pr(\gamma_{\mathrm{SD}} < \gamma_{\mathrm{SD},0}) p_{o,r}^{r_a - r_b} \\ &\quad \times \sum_{j = r_b - \tilde{r}_c(r_a) + 1}^{r_b} \binom{r_b}{j} p_{o,r}^j (1 - p_{o,r})^{r_b - j} && \text{(23a)} \\ &= \Theta(\bar{\gamma}^{-1}) \Theta(\bar{\gamma}^{-(r_a - r_b)}) \Theta(\bar{\gamma}^{-(r_b - \tilde{r}_c(r_a) + 1)}) && \text{(23b)} \\ &= \Theta(\bar{\gamma}^{-(r_a - \tilde{r}_c(r_a) + 2)}). && \text{(23c)} \end{aligned}$$

Similarly, let γ_{nec} be the maximum $\gamma_{\mathrm{SR},i}$ or $\gamma_{\mathrm{RD},i}$ for any relay i, such that if the system in outage, then $\gamma_{\mathrm{SD}} \leq \gamma_{\mathrm{SD},0}$ and $\gamma_{\mathrm{SR},i} < \gamma_{\mathrm{nec}}$ or $\gamma_{\mathrm{RD},i} < \gamma_{\mathrm{nec}}$ for at least $r_b - \tilde{r}_c(r_a) + 1$ of the r_b chosen relays. The analysis of the necessary condition is similar to (23a), with $\leq$ in place of $\geq$, and γ_{nec} in place of γ_{suf} in (22). This shows that for $r_b \geq \tilde{r}_c(r_a)$, the diversity order of the system is $r_a - \tilde{r}_c(r_a) + 2$. ■

REFERENCES

[1] T. M. Cover and A. A. El Gamal, "Capacity theorems for the relay channel," *IEEE Trans. on Inform. Theory*, vol. IT-25, no. 5, pp. 572–584, Sept. 1979.

[2] A. Chakrabarti, A. Sabharwal, and B. Aazhang, "Sensitivity of achievable rates for half-duplex relay channel," in *Proc. IEEE Workshop on Sig. Proc. Advances in Wireless Comm.*, New York, NY, June 2005.

[3] G. Kramer, M. Gastpar, and P. Gupta, "Cooperative strategies and capacity theorems for relay networks," *IEEE Trans. on Inform. Theory*, vol. 51, no. 9, pp. 3037–3063, Sept. 2005.

[4] P. Gupta and P. R. Kumar, "The capacity of wireless networks," *IEEE Trans. on Inform. Theory*, vol. 46, no. 2, pp. 388–404, Mar. 2000.

[5] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. on Inform. Theory*, vol. 50, no. 12, pp. 3062–3080, Dec. 2004.

[6] K. Azarian, H. El Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in half-duplex cooperative channels," *IEEE Trans. on Inform. Theory*, vol. 51, no. 12, pp. 4152–4172, Dec. 2005.

[7] J. Castura and Y. Mao, "Rateless coding for wireless relay channels," *IEEE Trans. on Wireless Comm.*, vol. 6, no. 5, pp. 1638–1642, May 2007.

[8] P. Razaghi and W. Yu, "Bilayer low-density parity-check codes for decode-and-forward in relay channels," *IEEE Trans. on Inform. Theory.*, vol. 53, no. 10, pp. 3723–3739, Oct. 2007.

[9] X. Bao and J. Li, "Efficient message relaying for wireless use cooperation: Decode-amplify-forward (DAF) and hybrid DAF and coded-cooperation," *IEEE Trans. on Wireless Comm.*, vol. 6, no. 11, pp. 3975–3984, Nov. 2007.

[10] H. H. Sneessens and L. Vandendorpe, "Soft decode and forward improves co-operative communcations," in *Proc. IEE Int. Conf. 3G and Beyond*, London, UK, Nov. 2005.

[11] J. P. K. Chu and R. S. Adve, "Implementation of co-operative diversity using message-passing in wireless sensor networks," in *Proc. IEEE Globecom*, St. Louis, MO, pp. 1167–1171, Dec. 2005.

[12] D. Chen and J. N. Laneman, "Modulation and demodulation for cooperative diversity in wireless systems," *IEEE Trans. on Wireless Comm.*, vol. 5, no. 7, pp. 1785–1794, Jul. 2006.

[13] T. R. Oenning and J. Moon, "A low density generator matrix interpretation of parallel concatenated single bit parity codes," *IEEE Trans. on Magnetics*, vol. 37, no. 2, pp. 737–741, Mar. 2001.

[14] D. Divsalar, H. Jin, and R. J. McEliece, "Coding theorems for 'turbo-like' codes," in *Proc. Allerton Conf. on Communications, Control, and Computing*, Monticello, IL, pp. 201–210, Sep. 1998.

[15] A. W. Eckford, J. P. K. Chu, and R. Adve, "Low-complexity and fractional coded cooperation for wireless networks," *IEEE Trans. on Wireless Comm.*, vol. 7, no. 5, pp. 1917–1929, May 2008.

[16] J. P. K. Chu, R. S. Adve, and A. W. Eckford, "Fractional cooperation using coded demodulate-and-forward," in *Proc. IEEE Globecom*, Washington, DC, pp. 4323–4327 Nov. 2007.

[17] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Trans. on Inform. Theory*, vol. 47, no. 2, pp. 599–618, Feb. 2001.

[18] S. Lin and D. J. Costello, *Error Control Coding: Fundamentals and Applications, 2nd Ed.*, Upper Saddle River, NJ: Prentice Hall, 2004.

[19] J. P. K. Chu, R. S. Adve, and A. W. Eckford, "Relay selection for low-complexity coded cooperation," in *Proc. IEEE Globecom*, Washington, DC, pp. 3932–3936, Nov. 2007.

[20] A. Bletsas, A. Khisti, D. Reed, and A. Lippman, "A simple cooperative diversity method based on network path selection," *IEEE Journal on Selected Areas in Communications*, vol. 24, no. 3, pp. 659–672, Mar. 2006.

[21] E. Beres and R. S. Adve, "Selection cooperation in multi-source co-operative networks," *IEEE Trans. on Wireless Comm.*, vol. 7, no. 1, pp. 118–127, Jan. 2008.

[22] Y. Zhao, R. S. Adve, and T. J. Lim, "Improving amplify-and-forward relay networks: Optimal power allocation versus selection," in *Proc. IEEE Int. Symp. on Information Theory*, Seattle, WA, pp.1234–1238, Jul. 2006.

[23] M. Ardakani and F. R. Kschischang, "A more accurate one-dimensional analysis and design of irregular LDPC codes," *IEEE Trans. on Communications*, vol. 52, no. 12, pp. 2106–2114, Dec. 2004.

[24] S. B. Qaisar and H. Radha, "Optimal progressive error recovery for wireless sensor networks using irregular low density parity check codes," in *Proc. Conf. on Inform. Sciences and Systems*, Baltimore, MD, pp. 232–237, Mar. 2007.

[25] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Trans. on Inform. Theory*, vol. 47, no. 2, pp. 498–519, Feb. 2001.

[26] A. Shokrollahi, "LDPC codes: An introduction." http://www.ipm.ac.ir/IPM/homepage/Amin2.pdf, 2003.

[27] M. Khojastepour, A. Sabharwal, and B. Aazhang, "On the capacity of 'cheap' relay networks," in *37th Annual Conf. on Inform. Sciences and Systems,* Baltimore, MD, Mar. 2003.

[28] S. Wei, D. Goeckel, and M. C. Valenti, "Asynchronous cooperative diversity," in *38th Annual Conf. on Inform. Sciences and Systems,* Princeton, NJ, 2004.

[29] A. Abbasfar, D. Divsalar, and K. Yao, "Accumulate-repeat-accumulate codes," *IEEE Trans. on Communications*, vol. 55, no. 4, pp. 692–702, Apr. 2007.

[30] R. J. McEliece, *The Theory of Information and Coding, 2nd ed.*, Cambridge: Cambridge University Press, 2002.

[31] R. Liu, P. Spasojević, and E. Soljanin, "Reliable channel regions for good binary codes transmitted over parallel channels," *IEEE Trans. on Inform. Theory*, vol. 52, no. 4, pp. 1405–1424, Apr. 2006.

[32] R. Liu, P. Spasojević, and E. Soljanin, "Incremental redundancy cooperative coding for wireless networks: Cooperative diversity, coding, and transmission energy gains," *IEEE Trans. on Inform. Theory*, vol. 54, no. 3, pp. 1207–1224, Mar. 2008.

[33] J. K. Omura and B. K. Levitt, "Coded error probability evaluation for antijam communication systems," *IEEE Trans. on Comm.*, vol. 30, no. 5, pp. 896–903, May 1982.

[34] R. Nikjah and N. C. Beaulieu, "On antijamming in general CDMA systems-part II: Antijamming performance of coded multicarrier frequency-hopping spread spectrum systems," *IEEE Trans. on Wireless Comm.*, vol. 7, no. 3, pp. 888–897, Mar. 2008.

[35] I. Sason and I. Goldenberg, "Coding for parallel channels: Gallager bounds and applications to turbo-like codes," *IEEE Trans. on Inform. Theory*, vol. 53, no. 7, pp. 2394–2428, Jul. 2007.

PLACE PHOTO HERE

Josephine P. K. Chu (S'98–M'99–S'01) received the B.Sc. degree in Electrical and Computer Engineering and the M.Sc. degree in Electrical Engineering from Queens University, Kingston, Ontario, Canada in 1999 and 2003 respectively. She is currently pursuing her Ph.D. at the Department of Electrical and Computer Engineering at the University of Toronto, Toronto, Ontario, Canada. Her research interests include communications and coding, with special emphasis on low-complexity relaying techniques.

PLACE PHOTO HERE

Raviraj S. Adve (S'88–M'97–SM'03) received his B. Tech. degree in Electrical Engineering from IIT, Bombay in 1990 and his Ph.D. from Syracuse University in 1996, where his dissertation received a Doctoral Prize. Between 1997 and August 2000, he worked for Research Associates for Defense Conversion Inc. on contract with the Air Force Research Laboratory at Rome, NY. He joined the faculty at the University of Toronto in August 2000. Dr. Adve's research interests include practical signal processing algorithms for smart antennas with applications in wireless communications and radar. He is currently focused on linear precoding in wireless communications, cooperative communications and augmenting space-time adaptive processing with a waveform dimension.

PLACE PHOTO HERE

Andrew W. Eckford (M'96–S'97–M'04) is originally from Edmonton, AB, Canada. He received the B.Eng. degree in electrical engineering from the Royal Military College of Canada, Kingston, ON, Canada, in 1996, and the M.A.Sc. as well as Ph.D. degrees in electrical engineering from the University of Toronto, Toronto, ON, Canada, in 1999 and 2004, respectively. Dr. Eckford is currently an Assistant Professor of Computer Science and Engineering at York University, Toronto, ON, Canada, a post he has held since July 2006.