Characterization of Relay Channels Using the Bhattacharyya Parameter

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***Abstract*—Relay systems have large and complex parameter spaces, which makes it difficult to determine the parameter region where the system achieves a given performance criterion, such as probability of frame error. In this paper, we show that the union bound (UB) and the Bhattacharyya parameter (BP) can be used for fast analysis of the parameter space when error-control coding is used. This is applicable when amplify-and-forward (AF) or demodulate-and-forward (DemF) are used. For a given code ensemble, the associated UB threshold is found and can be used to define the signal-to-noise region where a given frame error rate can be achieved. Using asymptotic results, the UB threshold can be used to specify the signal-to-noise ratio region where successful decoding can be achieved for large blocklength. In addition, the UB with BP can be used when fractional cooperation is used, where each relay only relays a fraction of the source codeword. This makes the UB with BP a valuable tool in the system design of relay networks.**

I. Introduction

In a relay channel, the transmission of a message from a source to a destination is aided by one or more intermediary nodes, known as relays [1]. The use of relays is known to improve performance in wireless fading channels, as it is improbable that all of the relays will simultaneously experience a deep fade. Some strategies for employing relays include *amplify-and-forward* (AF) [2], [3], in which the relay only performs analog amplification on the source's signal; *demodulate-and-forward* (DemF) [4], [5], in which the source's transmission is demodulated by a relay prior to retransmission; and *decode-and-forward* (DF) [2], [6], in which the relay decodes the source's codeword and re-encodes the information sequence.

The analysis of relay channels is complicated by their large parameter spaces. For instance, each additional relay adds at least two parameters: the signal-to-noise ratio (SNR) on the source-to-relay link, and the SNR on the relay-to-destination link (additional parameters may be introduced by the type of relaying in use). Thus, including the direct link from source to destination, an m-relay system has at least a $(2m+1)$-dimensional parameter space. Given a particular setting of all the parameters, it is not immediately clear whether a given performance criterion, such as frame error rate, would be satisfied by that setting. Furthermore, the large parameter space makes it difficult to characterize the entire space by simulation, density evolution, or any other method that examines the space at individual points.

Previous work has been done to characterize parameter spaces in related scenarios. In [7], the Union Bound-Bhattacharyya parameter (UB-BP) method was used to characterize the parameter space of parallel channels, where a different segment of the same codeword was transmitted on each channel (such a scenario may be found in a block fading channel). In [8], the BP was used for relay selection and outage probability analysis in relay channels employing DemF. In [9], the authors use the UB-BP method to find the SNR threshold for incremental redundancy (IR) where the decoding error becomes very small. It was also used to derive the diversity order of IR in fading channels.

The contribution in this present paper may be understood as an extension of these works, where with the use of the UB-BP method, a parameter space characterization of relay channels which employ AF and DemF is provided. Further, the formulation here provides a tool that can be used to compare AF and DemF. We show in Section IV that the use of the BP gives rise to an easily calculated figure of merit to verify whether a particular relay system satisfies a given performance criterion, here being the frame error rate (FER). We also give asymptotic results which can be used as a guide for more detailed density evolution analysis. Hence, the UB can be used by system designers to tweak the parameters of the relaying system, and hence determine the amount of relaying that is required to achieve given system criteria.

This paper is organized as follows. In Sec. II, we introduce the system model used throughout this paper and briefly describe the relaying schemes AF and DemF. Some background information on the union bound and the Bhattacharyya parameter is given in Sec. III. The application of UB and BP in relay channels is illustrated in Sec. IV. Simulation results are shown in Sec. V, and conclusions drawn in Sec. VI.

II. System Model

In this paper the m-relay network is used for the analysis, which includes the source (S), destination (D) and m relay (R_k) nodes. It is assumed that each relay can only transmit or receive at a time and that symbol synchronization is not available. Orthogonal channels are used to facilitate transmissions from different nodes. A quasi-static Rayleigh fading channel model is used to describe the links between the nodes. It is assumed that all receivers have channel state information, and the instantaneous SNR between S and all the relays are known at the destination as well.

The source node first forms a codeword $\mathbf{d}^{(S)} \in \{0,1\}^{l_s}$ of rate r_s, and assuming binary phase-shift keying (BPSK) is used, the codeword is mapped to $\mathbf{c}^{(S)} \in \{+1,-1\}^{l_s}$. This mapping is denoted by the function $\xi(\cdot)$. The source node broadcasts the codeword $\mathbf{c}^{(S)}$ and the discrete-time received signal at relay k and D are given by

$$\mathbf{y}_k^{(\mathrm{SR})} = h_{\mathrm{SR},k}\mathbf{c}^{(S)} + \mathbf{n}_k^{(\mathrm{SR})} \qquad k = 1,\dots,m \tag{1}$$

$$\mathbf{y}^{(\mathrm{SD})} = h_{\mathrm{SD}}\mathbf{c}^{(S)} + \mathbf{n}^{(\mathrm{SD})}, \tag{2}$$

where $h_{\mathrm{SR},k}$ and h_{SD} are the fading channel coefficients between S and the kth relay and between S and D respectively, and $\mathbf{n}_k^{(\mathrm{SR})}$ and $\mathbf{n}^{(\mathrm{SD})}$ are independent additive white Gaussian noise with variance $N_{R,k}$ and N_D respectively. After receiving $\mathbf{y}_k^{(\mathrm{SR})}$, relay k processes the received data, and forms a new symbol vector $\mathbf{c}_k^{(R)}$, which is then transmitted to the destination node. The length of the new signal formed by different relays can be different from that of the source. The discrete time signal transmitted by relay k and received by D is given by

$$\mathbf{y}_k^{(\mathrm{RD})} = h_{\mathrm{RD},k}\mathbf{c}_k^{(R)} + \mathbf{n}_k^{(\mathrm{RD})}, \qquad k = 1,\dots,m, \tag{3}$$

where $h_{\mathrm{RD},k}$ is the channel coefficient between relay k and D, and $\mathbf{n}_k^{(\mathrm{RD})}$ is the additive white Gaussian noise with variance N_D. The average received signal-to-noise ratio (SNR) of the channel between S and relay k is given by

$$\bar{\gamma}_{\mathrm{SR},k} = \mathbb{E}[\gamma_{\mathrm{SR},k}] = \mathbb{E}[|h_{\mathrm{SR},k}|^2]/N_{R,k}, \tag{4}$$

where $\mathbb{E}[\cdot]$ denotes statistical expectation and $\gamma_{\mathrm{SR},k}$ is the instantaneous SNR of the channel between S and the kth relay. The average received SNR of the S-D and R_k-D channel, $\bar{\gamma}_{\mathrm{SD}}$ and $\bar{\gamma}_{\mathrm{RD},k}$, can be found in a similar fashion.

A. Amplify-and-Forward

Recall that for AF, each relay R_k amplifies the received the signal and transmits it to D. The multiplication factor can be determined by either the amplitude limit imposed by the transmitting antenna, or the average transmitted power. In this paper, we will assume that the average transmitted power constraint is enforced, where the amplification factor is set such that the SNR over the channel between relay k and D is constant for each block.

Instead of relaying the complete source codeword, each relay can also relay only a *fraction* of the codeword. With *fractional cooperation* [10], each relay chooses at random a fraction of the source codeword to relay, where no centralized coordination is required. Here we assume that relay k chooses to relay each bit with probability ϵ_k. Let the ith bit of the source codeword be relayed as the $b_{k,i}$th bit of the symbol vector transmitted by the kth relay. Then

$$c_{k,b_{k,i}}^{(R)} = \frac{y_{k,i}^{(\mathrm{SR})}}{\sqrt{|h_{\mathrm{SR},k}|^2 + N_{R,k}}}$$

At the destination node, the received signal from the relays are combined with the received signal from the source node, each of them scaled accordingly to reflect channel conditions and the noise variance.

B. Demodulate-and-Forward

Cooperative DemF schemes are devised with sensor networks in mind, where the battery power and hardware complexity of a source or relay node are limited, but the destination is assumed to possess relatively more complex hardware, and energy consumption is not an issue. DemF differs from DF as it only requires symbol-by-symbol hard detection, and the relay is not required to decode the complete codeword. In addition, with DF, the signal is only relayed if correct decoding is achieved, whereas with DemF, the signals are always forwarded.

If DemF is used, after receiving the signal $\mathbf{y}_{\mathrm{SR},k}$, relay k makes hard decisions on the received signal to form $\mathbf{d}_k^{(R)}$. Similar to AF, fractional cooperation can be used so that the relay can select only part of the received signal to relay. In this case the new codeword $\mathbf{d}_k^{(R)}$ will only contain bits that are selected. It then generates $\mathbf{c}_k^{(R)} = \xi(\mathbf{d}_k^{(R)})$, which is then transmitted to D.

The decoding process uses sum-product algorithm (SPA) [11] to account for the reliability of the S-R channel. More information regarding the decoding process can be found in [10].

III. Union Bound and Bhattacharyya Parameter

The union bound can be used to provide upper bounds on the maximum likelihood (ML) bit error rate (BER) and frame error rate (FER) for linear codes [12]. Two components are required for calculating the union bound: the weight enumerator (WE) and the Bhattacharyya parameter (BP). The WE is a vector of numbers A_h that describes the number of codewords with weight h. The derivation of WE for "turbo-like" codes was presented in [13], where RA codes were also introduced. The second component, BP, is associated with the channel condition. The BP for binary input channels with inputs $\{0,1\}$ and output y is given by

$$\beta \triangleq \sum_{y\in\mathcal{Y}} \sqrt{p(y|0)p(y|1)} \tag{5}$$

where $\mathcal{Y}$ is the alphabet of output y, and $p(y|0)$ and $p(y|1)$ are, respectively, the probability of y given 0 and 1 was sent. For the binary symmetric channel (BSC) with bit-flip probability p, $\beta = 2\sqrt{p(1-p)}$, and for the additive white Guassian noise channel with received SNR γ, $\beta = e^{-\gamma}$. After deriving the WE and BP, the FER bound for a point-to-point channel is given by

$$P_f \leq \sum_{h=1}^{n} A_h \beta^h. \tag{6}$$

These results can be extended to scenarios where the codeword is sent through multiple parallel channels. The scenario where the codeword is divided into subsets, with each subset is transmitted through a different channel was analyzed in [7], where each channel can have a different BP. By averaging over all possible channel assignments for the codeword bits, the authors derived the parallel-channel UB on the FER. Assuming that there are J parallel channels, then

$$P_f \leq \sum_{h=1}^{n} A_h \left(\sum_{j=1}^{J} \alpha_j \beta_j \right)^h \tag{7}$$

where α_j is the probability that each bit of the codeword transmitted through channel j and $\sum_{j=1}^{J} \alpha_j = 1$ and β_j is the BP for channel j. The IR cooperative coding scheme, where the source codeword is divided into subsets, with each subset relayed by at most one relay, is analyzed using UB-BP in [9].

For codewords with large block lengths, asymptotic results can be used to estimate the SNR threshold above which the error rate decreases very quickly. Let $[\mathcal{C}(n)]$ denote a binary code ensemble with rate r and length n, and let $A_h^{[\mathcal{C}(n)]}$ denote the average number of codewords with Hamming weight h for the entire code ensemble $[\mathcal{C}(n)]$. Then the asymptotic normalized exponent of the weight spectrum is defined as [9]

$$r^{[\mathcal{C}]}(\delta) \triangleq \limsup_{n\to\infty} \frac{\ln A_h^{[C(n)]}}{n}$$

where $[\mathcal{C}]$ denotes the binary code ensemble with rate r. Using the same notation as [9], we define the *asymptotic UB threshold* of a code ensemble $[\mathcal{C}]$ as

$$c_0 \triangleq \sup_{0<\delta\leq 1} \frac{r(\delta)}{\delta}, \tag{8}$$

With some manipulation of (6), it can be shown that if

$$\sum_{j=1}^{J} \alpha_j \beta_j < \exp(-c_0) \tag{9}$$

then the average FER approaches zero as $n \to \infty$.

Note that this section only provides a brief overview of the union bound and BP. A more detailed description, including the derivation of the union bound, can be found in [12], and discussions of tighter decoding error bounds can be found in [7], [14], [15].

IV. Application of UB-BP to Relay Channels

Similar to the single-input single-output channels, the UB and BP can be applied to relay channels. We will first show the BP corresponding to two different schemes where repetition code is used: AF and DemF. In both cases, the relay repeats data based on the received signals, and no coding is performed at the relay. We will then show how the UB-BP can be used in relay channels to determine the region in the parameter space where a given performance criterion can be met when AF or DemF is used.

A. BP for AF Repetition Code

Recall that with the use of fractional cooperation, each bit can be relayed by any number of relays ranging from 0 to m. Let M_i be the set of relays that are relaying bit i of the source codeword. After combining the received signals from the source and all the relays, we have the ith bit of the combined signal $\mathbf{y_d}$

$$y_{d,i} = \frac{h_{\mathrm{SD}}^* y_i^{(\mathrm{SD})}}{N_D} + \sum_{k\in M_i} \frac{h_{\mathrm{SR},k}^* h_{\mathrm{RD},k}^* y_{k,b_{k,i}}^{(\mathrm{RD})}}{N_{\mathrm{SRD},i}\sqrt{|h_{\mathrm{SR},k}|^2 + N_{R,k}}} \tag{10}$$

where x^* is the conjugate of x and $N_{\mathrm{SRD},k} = N_D + \frac{|h_{\mathrm{RD},k}|^2 N_{R,k}}{|h_{\mathrm{SR},k}|^2+N_{R,k}}$. After some manipulation, it can be shown that $y_{d,i}$ is a Gaussian signal, with SNR given by

$$\gamma_{\mathrm{AF},i} = \gamma_{\mathrm{SD}} + \sum_{k\in M_i} \frac{\gamma_{\mathrm{SR},k}\gamma_{\mathrm{RD},k}}{\gamma_{\mathrm{SR},k} + \gamma_{\mathrm{RD},k} + 1} \tag{11}$$

Recall that for a Gaussian channel with SNR γ, the BP is given by $\beta = \exp\{-\gamma\}$. Hence the BP of bit i for the AF coding scheme is given by

$$\beta_{\mathrm{AF}} = \exp\left\{-\gamma_{\mathrm{SD}} - \sum_{k\in M_i} \frac{\gamma_{\mathrm{SR},k}\gamma_{\mathrm{RD},k}}{\gamma_{\mathrm{SR},k} + \gamma_{\mathrm{RD},k} + 1}\right\} \tag{12}$$

$$= \beta_{\mathrm{SD}} \prod_{k\in M_i} \beta_{\mathrm{AF},k} \tag{13}$$

where

$$\beta_{\mathrm{SD}} = \exp\{-\gamma_{\mathrm{SD}}\}$$
$$\beta_{\mathrm{AF},k} = \exp\left\{-\frac{\gamma_{\mathrm{SR},k}\gamma_{\mathrm{RD},k}}{\gamma_{\mathrm{SR},k} + \gamma_{\mathrm{RD},k} + 1}\right\} \tag{14}$$

In the case of AF, the SNR of the equivalent channel may be exactly calculated, as in (11). By placing it in the UB-BP expression (12), and noting that this expression can be used to verify that a given performance criterion is satisfied, $\beta_{\mathrm{AF},k}$ – or the equivalent expression in any other relaying system – may be used as a *figure of merit* for that system.

B. BP for DemF Repetition Code

For DemF, we have the following likelihood equations for the ith received signal from relay k

$$\begin{aligned} p(y_{k,i}^{(\mathrm{RD})}|c_i^{(S)} = +1) &= (1 - p_{\mathrm{SR},k}) f(y_{k,i}^{(\mathrm{RD})}; 1, 1/2\gamma_{\mathrm{RD},k}) \\ &\quad + p_{\mathrm{SR},i} f(y_{k,i}^{(\mathrm{RD})}; -1, 1/2\gamma_{\mathrm{RD},k}) \\ p(y_{k,i}^{(\mathrm{RD})}|c_i^{(S)} = -1) &= (1 - p_{\mathrm{SR},k}) f(y_{k,i}^{(\mathrm{RD})}; -1, 1/2\gamma_{\mathrm{RD},k}) \\ &\quad + p_{\mathrm{SR},k} f(y_{k,i}^{(\mathrm{RD})}; 1, 1/2\gamma_{\mathrm{RD},k}) \end{aligned}$$

where

$$p_{\mathrm{SR},k} = \frac{1}{2}\mathrm{erfc}(\sqrt{\gamma_{\mathrm{SR},k}})$$
$$f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(y-\mu)^2}{2\sigma^2}\right\}$$

After substituting these equations into (5) and some simple manipulations, we have

$$\begin{aligned} \beta_{\mathrm{DemF},k} = \sqrt{\frac{\gamma_{\mathrm{RD},k}}{\pi}} \int_{-\infty}^{\infty} & \exp\left\{-\gamma_{\mathrm{RD},k}(y^2+1)\right\} \\ & [(1 - p_{\mathrm{SR},k} + p_{\mathrm{SR},k}\exp\left\{-\gamma_{\mathrm{RD},k} y\right\}) \\ & (1 - p_{\mathrm{SR},k} + p_{\mathrm{SR},k}\exp\left\{\gamma_{\mathrm{RD},k} y\right\})]^{1/2}\, dy \end{aligned} \tag{15}$$

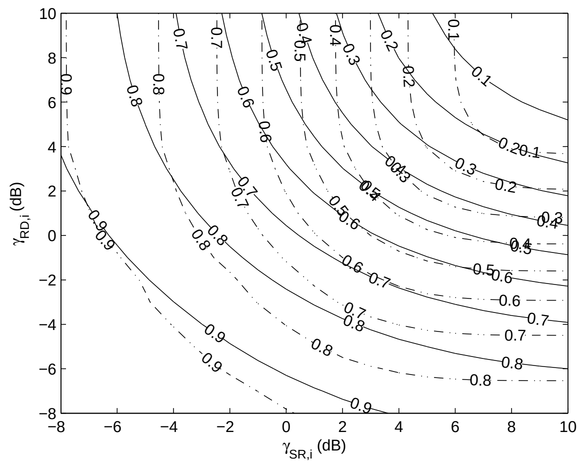

Fig. 1. Contours of β_i for AF (solid) and DemF (dash-dot).

A closed form solution to the integration in (15) does not exist, but the integral can be approximated using the Taylor expansion of $\sqrt{1+x}$ for $|x| < 1$. The received signal from the relays are combined with the signal from S in the decoding process. Similar to the AF repetition code, if fractional cooperation is used, then we have the BP associated with ith bit of the equivalent received signal given by

$$\beta_{\mathrm{DemF},i} = \beta_{\mathrm{SD}} \prod_{k \in M_i} \beta_{\mathrm{DemF},k}. \tag{16}$$

This quantity may be used as a figure-of-merit for a DemF system, which is significant since the equivalent SNR of DemF cannot be calculated as easily as for AF.

In both AF and DemF, the BP "contribution" for each relay can be isolated. The $\beta_{\mathrm{AF},k}$ and $\beta_{\mathrm{DemF},k}$ values are shown in Fig. 1. The contours of various β values are plotted for different $\gamma_{\mathrm{SR},k}$ and $\gamma_{\mathrm{RD},k}$ values. In the figure, the solid lines represent values of $\beta_{\mathrm{AF},k}$ and the dash-dot lines represent values of $\beta_{\mathrm{DemF},k}$. In most cases, $\beta_{\mathrm{DemF},k} < \beta_{\mathrm{AF},k}$, except when $\gamma_{\mathrm{RD},k}$ is much larger than $\gamma_{\mathrm{SR},k}$, as indicated by the crossover points in the plot. The BP can be used to compare the performance the AF and DmF, and helps us estimate under what condition it is more advantageous to use AF or DmF. If a relay is capable of both AF and DmF, then it can choose the relaying scheme that will provide the best error performance given the channel conditions.

C. Decodable region

The upper bounds on the FER can be used to specify the region in the $(2m+1)$-dimensional SNR space where a given error rate can be achieved. Depending on the relaying scheme, we can then substitute (14) or (15) into (7) to obtain the upper bound on the error performance. Let $I = \{1, 2, \ldots, m\}$ be the set of relays, and I_j be all disjoint subsets of I, including the empty subset $\emptyset$, $j = 1, \ldots, 2^m$. Then the UB is given by

$$P_f \le \sum_{h=1}^{n} A_h \left(\beta_{\mathrm{SD}} \sum_{j=1}^{2^m} \alpha_j \left(\prod_{k \in I_j} \beta_k \right) \right)^h \tag{17}$$

where

$$\alpha_j = \left(\prod_{k \in I_j} \epsilon_k \right) \left(\prod_{k \in I \setminus I_j} 1 - \epsilon_k \right)$$

and β_k can be $\beta_{\mathrm{AF},k}$ or $\beta_{\mathrm{DemF},k}$, depending the type of relaying scheme chosen by relay k. With some simple manipulations, (17) can be simplified to

$$P_f \le \sum_{h=1}^{n} A_h \left(\beta_{\mathrm{SD}} \prod_{k=1}^{m} (1 - \epsilon_k (1 - \beta_k)) \right)^h \tag{18}$$

To find the region of SNR where the FER is below $P_{f,t}$, we need to first find the value of β_t such that

$$P_{f,t} = \sum_{h=1}^{n} A_h \beta_t^h.$$

Then it is easy to see that as long as the condition

$$\beta_{\mathrm{SD}} \prod_{k=1}^{m} (1 - \epsilon_k (1 - \beta_k)) \le \beta_t \tag{19}$$

is satisfied, the FER is below $P_{f,t}$. The set of SNRs where the condition in (19) is satisfied defines the region in the $(2m+1)$-dimensional SNR space where the FER requirement is fulfilled.

For source codewords with asymptotically large blocklength, if the asymptotic UB threshold c_0 exists and is finite, it can be used to estimate the SNR region where the FER approaches zero [7]. It can be assumed that successful decoding can be achieved if the condition

$$\beta_{\mathrm{SD}} \prod_{k=1}^{m} (1 - \epsilon_k (1 - \beta_k)) \le \exp(-c_0) \tag{20}$$

is satisfied. As a result, the expression in (20) can be used to predict the outcomes of density evolution.

V. Simulation Results

This section compares Monte Carlo simulation results and the associated union bounds. For all the results shown here, a rate-1/2 systematic RA code is used, and the blocklength of the source codeword, n is 16000. It is assumed that the information bits are repeated $q = 3$ times and punctured to form the parity bits [16]. The WE can be obtained using the formulation in [17]. In Fig. 2, only one relay is used, and the SNR over all channels are the same. Simulation results for both AF and DemF are shown, and the cases where $\epsilon_k = 1$ and $\epsilon_k = 1/2$ are illustrated. The SNR thresholds corresponding to the asymptotic UB threshold c_0 of the rate-1/2 systematic RA code for various cases are also illustrated. In all the cases,

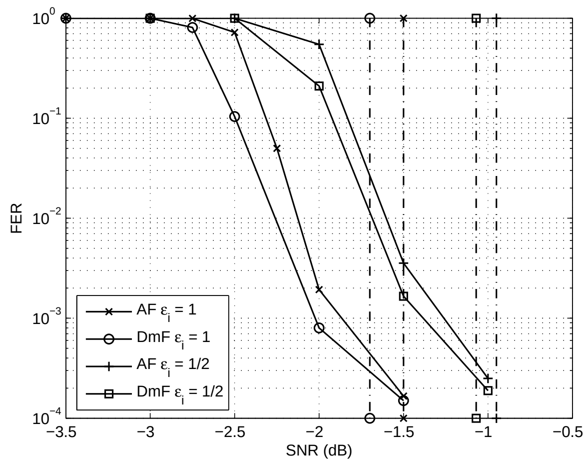

Fig. 2. FER for AF and DemF with blocklength n = 16000 and their corresponding SNR thresholds derived from c_0 of the rate-1/2 RA code.

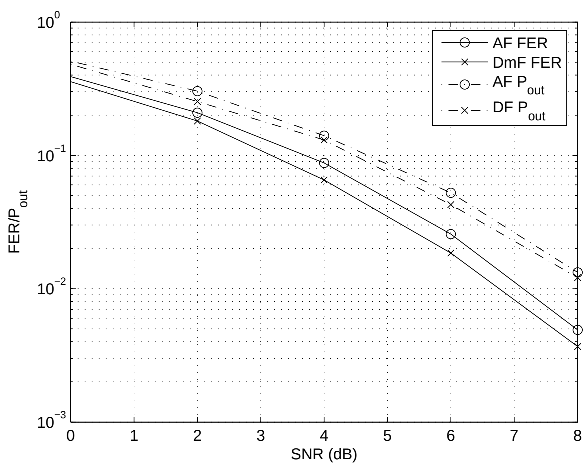

Fig. 3. FER for AF and DemF with blocklength n = 16000 and P_{out} of the rate-1/2 RA code with $m = 6$ and $\epsilon_i = 0.2$.

the waterfall of the FER is about 0.5 dB away from the SNR threshold corresponding to c_0.

In Fig. 3, the FER for multiple relays in fading channels are shown. In this case, $m = 6$, $\epsilon_i = 0.2$ and the average SNR is the same over all channels. The probability of (20) not satisfied, or equivalently, the outage probability P_{out} for both AF and DemF are also shown in the plot. As shown in the plot, P_{out} follows the FER closely. Although the outage probability only provides an upper bound on the FER, and does not provide an exact expression for the FER, it can be used to gain an understanding of the diversity order of the system. Similar to the previous scenario, DmF performs better than AF when the average SNR over all the channels are the same.

VI. Conclusion

In addition to single-in single-out channels, UB-BP can also be used for analysis in relay channels. In this paper, we have presented the BP associated with AF and DemF repetition code. We have also shown how the UB with BP can be used to determine the SNR region where a given error performance can be achieved. By applying asymptotic results, it can be used to determine where correct decoding is achievable for large blocklengths. Finally, we have shown that this is not only applicable to relay channels with one relay, but also ones with multiple relays with fractional cooperation.

References

[1] T. M. Cover and A. A. E. Gamal, "Capacity theorems for the relay channel," *IEEE Trans. on Inform. Theory*, vol. IT-25, pp. 572–584, Sept. 1979.

[2] J. N. Laneman, D. N. C. Tse, and G. W. Wornell, "Cooperative diversity in wireless networks: Efficient protocols and outage behavior," *IEEE Trans. Inform. Theory*, vol. 50, pp. 3062–3080, Dec. 2004.

[3] Y. Zhao, R. Adve, and T. Lim, "Improving amplify-and-forward relay networks: Optimal power allocation versus selection," *IEEE Trans. on Wireless Comm.*, vol. 6, pp. 3114–3123, Aug. 2007.

[4] J. P. K. Chu and R. S. Adve, "Implementation of co-operative diversity using message-passing in wireless sensor networks," in *Proc. IEEE Globecom*, Dec. 2005.

[5] T. Wang, A. Cano, G. B. Giannakis, and J. N. Laneman, "High-performance cooperative demodulation with decode-and-forward relays," *IEEE Trans. on Comm.*, vol. 55, pp. 1427–1438, July 2007.

[6] P. Razaghi and W. Yu, "Bilayer low-density parity-check codes for decode-and-forward in relay channels," *IEEE Trans. Inform. Theory*, vol. 53, pp. 3723–3739, Oct. 2007.

[7] R. Liu, P. Spasojević, and E. Soljanin, "Reliable channel regions for good binary codes transmitted over parallel channels," *IEEE Trans. on Inform. Theory*, vol. 52, pp. 1405–1424, April 2006.

[8] J. P. K. Chu, R. S. Adve, and A. W. Eckford, "Using the Bhattacharyya parameter for design and analysis of cooperative wireless systems." Submitted to *IEEE Trans. Wireless Comm.*, under review.

[9] R. Liu, P. Spasojević, and E. Soljanin, "Incremental redundancy cooperative coding for wireless networks: Cooperative diversity, coding, and transmission energy gains," *IEEE Trans. on Inform. Theory*, vol. 54, pp. 1207–1224, Mar. 2008.

[10] A. W. Eckford, J. P. K. Chu, and R. Adve, "Low-complexity and fractional coded cooperation for wireless networks," *IEEE Trans. on Wireless Comm.*, vol. 7, pp. 1917–1929, May 2008.

[11] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Trans. Inform. Theory*, vol. 47, pp. 498–519, Feb. 2001.

[12] S. Lin and D. J. Costello, *Error Control Coding: Fundamentals and Applications*. Prentice Hall, second ed., 2004.

[13] H. Jin and R. J. McEliece, "RA codes achieve AWGN channel capacity," in *Proc. 13th Int. Symp. on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*, pp. 10–18, Nov. 1999.

[14] S. Shamai (Shitz) and I. Sason, "Variations on the gallager bounds, connections, and applications," *IEEE Trans. on Comm.*, vol. 48, pp. 3029–51, December 2002.

[15] I. Sason and I. Goldenberg, "Coding for parallel channels: Gallager bounds and applications to turbo-like codes," *IEEE Trans. on Inform. Theory*, vol. 53, pp. 2394–2428, July 2007.

[16] D. Divsalar, H. Jin, and R. J. McEliece, "Coding theorems for "turbo-like" codes," in *Proc. Allerton Conf. on Comm., Control, and Computing*, pp. 210–210, Sept. 1998.

[17] A. Abbasfar, D. Divsalar, and K. Yao, "Maximum likelihood decoding analysis of accumulate-repeat-accumulate codes," in *Proc. IEEE Globecom*, Dec. 2004.