Analysis of Random Radar Networks

Rani Daher, Raviraj Adve
Department of Electrical and Computer Engineering, University of Toronto
10 King's College Road, Toronto, ON M5S3G4
Email: rani.daher@utoronto.ca, rsadve@comm.utoronto.ca

***Abstract*—We introduce the notion of *random* radar networks to analyze the effect of geometry in distributed radar systems. We first analyze unistatic systems with a single receiver selected at random from the available group. We approximate the distribution of the individual Signal-to-Interference-Plus-Noise (SINR) at the sensors and find the corresponding mean and variance. We then analyze multistatic systems and provide an upper bound on performance. We show that in order to exploit the spatial diversity available to the system, the sensors should be large enough to effectively cancel interfering sources. We underline a design tradeoff between spatial diversity and interference cancellation for multistatic radar networks. We finally provide the results of simulations to validate our analysis.**

I. Introduction

Recently, detection using distributed radar apertures has received renewed attention. Such a system avails the benefits of *spatial diversity*. However, there has been little theoretical analysis to understand the trade-offs involved in distributing sensors. In our previous work [1], we introduced the notion of the *diversity order* of a noise-limited distributed radar system. This notion allows a system designer to evaluate the trade-off between co-located and distributed sensors. We proved that larger antenna arrays are preferable for noise-limited systems, and that most of the performance can be achieved with a limited communication bandwidth between the sensors and the fusion center. However, more practical systems include interference, which is, generally, a function of the geometry of the system.

In [2], Goodman *et. al.* initiated the work on geometry. They proved empirically that networks with certain geometries are able to combat interference more effectively that other geometries. The drawback is that the available literature deals with systems with fixed geometries. The main contribution of this paper is the introduction of the notion of a *random network*. This model is relevant because the relative directions and velocities of the target cannot be known *a-priori*. Furthermore, even the locations of the sensors are generally not regular. In this work we propose a formal approach to tackle this inherent randomness of the network.

Specifically, in this paper, our objective is to determine the interaction between spatial diversity and the required number of antennas at each sensor for effective interference cancellation. Here, we use the output signal-to-interference-plus-noise ratio (SINR) as the metric for system performance. The network being random, the SINR is in turn a random variable. We first determine the distribution of the SINR, and we then determine its corresponding mean and variance. We begin by analyzing a unistatic system with a single sensor chosen randomly from an available group of sensors.

For random multistatic networks, we recur to the definition of diversity that we first introduced in [1]. We show the existence of a tradeoff: the sensors should be large enough to cancel the interference, and at the same time, a larger number of sensors enables the exploitation of spatial diversity and achieves higher diversity order. System designers should find the proper balance between the size and the number of sensors to conform to resource limitations and abide to performance requirements. We finally provide results of simulations validating our theoretical analysis. We show the results for joint optimal detection and for fully distributed systems using the OR rule.

This paper is organized as follows: Section II introduces the system model under consideration and provides a brief background on distributed detection and the Neyman-Pearson (NP) test. In Section III we analyze a unistatic system, and derive the characteristics of the SINR for a single sensor. In Section IV we analyze multistatic systems and provide analytical and empirical evaluations of spatial diversity in distributed STAP systems. The paper is concludes with some suggestions for future work in Section V.

II. System Model and Background

In this section we present our system model and a brief overview of the available literature on distributed detection and interference cancellation.

A. System Model

Detection is performed with K distributed sensors attempting to detect the presence of a target in a certain region in space. Each sensor possesses N collocated antennas. The k-th sensor receives a data vector of the form:

$$\mathbf{z}_k = \begin{cases} \alpha_k \mathbf{s}_k + \mathbf{n}_k, & \text{if target is present,} \\ \mathbf{n}_k, & \text{if target is absent,} \end{cases} \tag{1}$$

where $\mathbf{s}_k$ is the target spatial steering vector corresponding to the target look direction, α_k is the complex-valued amplitude, and $\mathbf{n}_k$ is the additive interference and noise. The target is modeled as a Swerling type-II and, consequently, $\{\alpha_k\}_{k=1}^{K}$ are independent and identically distributed (i.i.d.) drawn from a

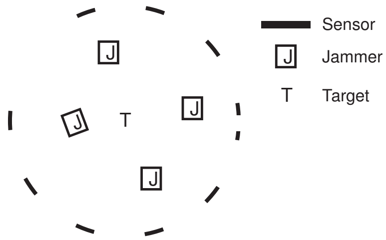


Fig. 1. Generic Model.

zero-mean complex Gaussian random variable whose variance determines the received signal-to-noise ratio (SNR).

In this work, we assume that, given the hypothesis, the observations at the sensors are statistically independent. Each sensor k transmits a decision u_k to a fusion center, which makes the final decision u_0 indicating the presence (hypothesis H_1) or absence (hypothesis H_0) of a target in the region of space monitored by the sensors. **u** is the length-K vector of the decisions of the sensors. We also assume that the noise statistics are known, and that fusion center receives the data from the local sensors without error.

Finally, we introduce our model of a *random* distributed network, a system where the relative positions and velocities of the sensors, target, jammers and clutter are all random. For reasons of practicality, we introduce a *generic* model (Figure 1) which assumes that the look region is at the center of the area monitored by the sensors, which in turn are *randomly* placed on a circle centered at the potential target; the jammers are randomly distributed inside this circle. A jammer is modeled as an interfering source originating from a single point in space. Note that this generic model is a special case of a random network where the restriction to a circle allows for the convenient symmetry assumption that the sensors receive equal SNR.

The optimal weight vector at receiver k that maximizes the output SINR is [3]:

$$\mathbf{w}_k = \mathbf{R}_{nk}^{-1}\mathbf{s}_k, \tag{2}$$

where $\mathbf{R}_{nk}$ is the interference-plus-noise covariance matrix, leading to the Neyman-Pearson statistic of

$$\zeta = \sum_{k=1}^{K} \frac{A^2|\mathbf{s}_k^H\mathbf{R}_n^{-1}\mathbf{z}_k|^2}{1 + A^2\mathbf{s}_k^H\mathbf{R}_n^{-1}\mathbf{s}_k}. \tag{3}$$

Performance greatly depends on the geometry of the system. For example, if the target is close in direction to a strong jammer, the target will be nulled. However, system designers have no control over this geometry. The main contribution of this paper, therefore, is an analysis of radar networks with random geometry. In the following section we will discuss unistatic random networks, and we will extend this analysis to multistatic systems in Section IV.

III. Random Unistatic Systems

When a certain range is monitored by *a single* sensor that is randomly chosen from the set of available sensors, we refer to the system as a random unistatic system. We assume that the sensor is randomly chosen without resorting to any pre-processing scheme that might assist the network in choosing the sensor that is best-fitted for detection. The general setting with an arbitrary number of interferers appears to be intractable for any significant analysis; we will therefore analyze a theoretical scenario where the system is limited by noise and a single *strong* jammer. We will derive the mean and variance of the output SINR. Since $K = 1$ in this case, in this section we drop the index k.

In the case of a single jammer, the noise-plus-interference covariance matrix is

$$\mathbf{R}_n = \sigma^2\left(\mathbf{I} + \gamma_j\mathbf{a}_j.\mathbf{a}_j^H\right), \tag{4}$$

where γ_j is the jammer-to-noise ratio (JNR) and $\mathbf{a}_j$ is the jammer steering vector. Using the matrix inversion lemma:

$$\mathbf{R}_n^{-1} = \frac{1}{\sigma^2}\left(\mathbf{I} - \frac{\gamma_j\mathbf{a}_j.\mathbf{a}_j^H}{1+\gamma_j\mathbf{a}_j^H\mathbf{a}_j}\right). \tag{5}$$

Note that $\mathbf{a}_j^H\mathbf{a}_j = \mathbf{s}^H\mathbf{s} = N$ where N is the number of antenna elements. The output SINR becomes:

$$\begin{aligned}\gamma_o &= |\alpha|^2\mathbf{s}^H\mathbf{R}_n^{-1}\mathbf{s}\\ &= \frac{|\alpha|^2}{\sigma^2}\left(N - \frac{\gamma_j|\mathbf{s}^H\mathbf{a}_j|^2}{1+N\gamma_j}\right)\\ &= \frac{|\alpha|^2}{\sigma^2}\left(N - \frac{\gamma_j|\sum_{n=0}^{N-1}e^{in\pi(\cos(\theta_t)-\cos(\theta_j))}|^2}{1+N\gamma_j}\right).\end{aligned} \tag{6}$$

In this scenario, the SINR is a function of two independent random variables: α and $u = [\cos(\theta_t) - \cos(\theta_j)]$. Under the Swerling type-II target assumption, α is complex Gaussian with average power $A^2 = \mathcal{E}\{|\alpha|^2\}$. The SINR expression simplifies using the fact that:

$$\begin{aligned}\mathcal{G}(u) &= \left|\sum_{n=0}^{N-1}e^{in\pi u}\right|^2 = \left(\sum_{n=0}^{N-1}\cos(n\pi u)\right)^2 + \left(\sum_{n=0}^{N-1}\sin(n\pi u)\right)^2\\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}[\cos(n\pi u)\cos(m\pi u) + \sin(n\pi u)\sin(m\pi u)]\\ &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\cos[(n-m)\pi u].\end{aligned} \tag{7}$$

Now define the all-ones vector $\mathbf{1}(N)$ of length N and define the length-$(2N-1)$ vector $\triangle$ as the linear convolution of $\mathbf{1}(N)$ with itself. In the sum of Eqn. (7), each term $j = (n-m) = -N+1, \ldots, N-1$ is repeated $\triangle(j+N)$ times. Therefore,

$$\begin{aligned}\mathcal{G}(u) &= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\cos[(n-m)\pi u]\\ &= N + 2\sum_{j=1}^{N-1}\triangle(j)\cos(j\pi u),\end{aligned} \tag{8}$$

where $\triangle(N) = N$ by virtue of the linear convolution.

A. Mean and Variance of the SINR

We now derive the mean and variance of the SINR. We first assume that the random variable u is uniform over the range $[-2, 2]$, i.e. $u \sim \mathcal{U}\{-2, 2\}$. The range is chosen so as to conform to the difference of 2 cosines. This is simply a tractable model to develop an understanding about the behavior of the mean and variance of the SINR.

1) Mean:: We start with the mean:

$$\mathcal{E}\{\mathcal{G}(u)\} = \int_{-2}^{2} \mathcal{G}(u) f_u(u) du = \frac{1}{4} \int_{-2}^{2} \left[N + 2 \sum_{j=1}^{N-1} \triangle(j) \cos(j\pi u) \right] du = N, \tag{9}$$

Using Eqn. (9) in Eqn. (6) we get:

$$\mathcal{E}\{\gamma_o\} = \frac{A^2}{\sigma^2} \left(N - \frac{N\gamma_j}{1 + N\gamma_j} \right), \tag{10}$$

which for large JNR, reduces to:

$$\mathcal{E}\{\gamma_o\} \approx \frac{A^2}{\sigma^2} (N - 1), \tag{11}$$

thus providing a theoretical explanation for the intuitive result that nulling a single jammer costs the system *on average* one degree of freedom.

2) Variance: To obtain the variance of the SINR, denote the second term of Eqn. (8) as:

$$\mathcal{Y} = 2 \sum_{j=1}^{N-1} \triangle(j) \cos(j\pi u). \tag{12}$$

Note that $\mathcal{E}\{\mathcal{Y}\} = 0$ and $\mathrm{var}\{\mathcal{Y}\} = \mathrm{var}\{\mathcal{G}(u)\}$ since N is a constant. We now derive the variance of $\mathcal{Y}$.

$$\begin{aligned} \mathrm{var}\{\mathcal{Y}\} &= \mathcal{E}\left\{ 4 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} \triangle(n)\triangle(m) \times \cos(n\pi u)\cos(m\pi u) \right\} \\ &= 4 \sum_{n=1}^{N-1} \sum_{m=1}^{N-1} \triangle(n)\triangle(m) \times \mathcal{E}\Big\{ \cos(n\pi u)\cos(m\pi u) \Big\}. \end{aligned}$$

Recalling that $u \sim \mathcal{U}\{-2, 2\}$ we get:

$$\mathcal{E}\Big\{ \cos(n\pi u)\cos(m\pi u) \Big\} = \frac{1}{4} \int_{-2}^{2} \cos(n\pi u)\cos(m\pi u) du = \begin{cases} \frac{1}{2}, & \text{if} \quad n = m \\ 0, & \text{if} \quad n \neq m \end{cases}, \tag{13}$$

and therefore

$$\mathrm{var}\{\mathcal{Y}\} = 2 \sum_{n=1}^{N-1} \triangle^2(n). \tag{14}$$

In what follows, we will denote the summation above by:

$$\mathcal{S}\{M\} = \sum_{n=1}^{M} \triangle^2(n) = 1^2 + 2^2 + \cdots + M^2, \quad M \leq N \tag{15}$$

$$= \frac{M(M+1)(2M+1)}{6}. \tag{16}$$

Let $\mathcal{Z} = \mathbf{s}^H \mathbf{R}_n^{-1} \mathbf{s}$ so that $\gamma_o = |\alpha|^2 \mathcal{Z}$. We assume without loss of generality that $A^2 = \mathcal{E}\{|\alpha|^2\} = 1$ and consequently, $\mathcal{E}\{|\alpha|^4\} = 2$ because $|\alpha|^2$ is exponentially distributed. This means that the input SNR is determined by the receiver noise variance σ^2. We first use Eqn. (6) to state that:

$$\begin{aligned} \mathrm{var}\{\mathcal{Z}\} &= \frac{1}{\sigma^4} \left(\frac{\gamma_j}{1 + N\gamma_j} \right)^2 \mathrm{var}\{\mathcal{Y}\} \\ &= \frac{2}{\sigma^4} \left(\frac{\gamma_j}{1 + N\gamma_j} \right)^2 \mathcal{S}\{N - 1\}. \end{aligned} \tag{17}$$

Simple computations using Eqns. (10) and (17) leads to:

$$\mathrm{var}\{\gamma_o\} = \frac{1}{\sigma^4} \left[\frac{4\mathcal{S}\{N-1\}\gamma_j^2}{(1 + N\gamma_j)^2} + \left(N - \frac{\gamma_j N}{1 + N\gamma_j} \right)^2 \right]. \tag{18}$$

For high JNR,

$$\begin{aligned} \mathrm{var}\{\gamma_o\} &\approx \frac{1}{\sigma^4} \left[\frac{4\mathcal{S}\{N-1\}}{N^2} + (N-1)^2 \right], \\ &= \frac{1}{\sigma^4} \left[\frac{(N-1)(N+1)(3N-2)}{3N} \right]. \end{aligned} \tag{19}$$

We note that the variance increases on the order of N^2 while the expectation is on the order of N. Clearly, therefore, a *single* receiver does not provide any gains in reliability

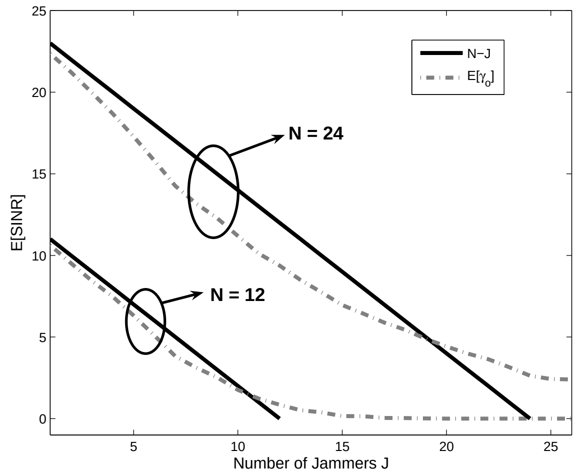
B. Numerical Results

This section presents the results of simulations to test the theory developed above. Figure 2(a) presents the average SINR for the generic model when the number of jammers J is varied. On the same graph, we also plot the straight line $y = N - J$. Figure 2(a) shows the mean SINR value when the number of interfering sources grows large, for $N = 12, 24$. In fact, when the sensor is not able to null the interference, the SINR per sensor drops to zero, and the system turns futile. For lower values of J, the SINR follows the linear curve predicted by Eqn. (11) extended to multiple jammers.

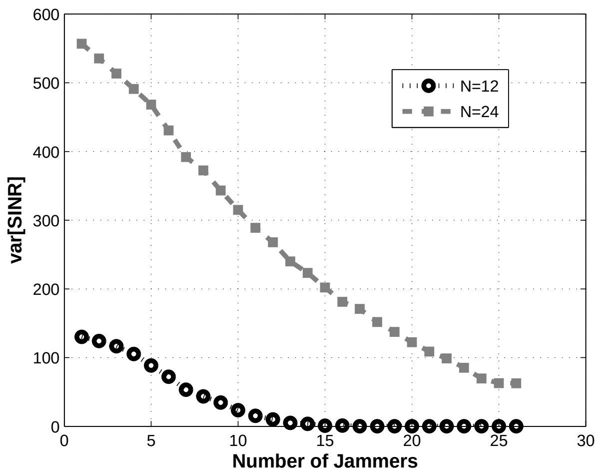
Figure 2(b) presents the variance for the same scenario. It shows that the variance is inversely proportional to the number of jammers. We note however that the mean also decreases and hence the reduction in the variance does not imply better performance. On the other hand, we notice a significant difference in the variance between $N = 12$ and $N = 24$; and the twofold difference in N is translated faithfully into a fourfold difference in the variance for the lowest values of J corroborating the analysis above.

IV. Random Multistatic Systems

The previous section focused on a single sensor with N collocated antennas. This sensor was picked randomly from a set of available sensors. We now analyze multistatic random radar networks. In such systems, K randomly located sensors detect the presence of a target at a certain range. There are two main system design problems: 1) how many sensors does reliable detection require?, and 2) how many antennas should each sensor possess?. We will start by stating an upper



(a) Mean of SINR



(b) Variance of SINR

Fig. 2. Mean and variance of the SINR with increasing number of jammers.

bound on performance. This upper bound illustrates the spatial diversity available due to distributed sensors. We then present alternative figures of merit that validate this analysis. It is worth emphasizing that an analytically tractable and consistent figure of merit has not yet been formulated for distributed detection.

A. An Upper Bound on SINR

In a system that adopts joint processing, i.e., when the data from each sensor is fully available at a fusion center, and under the assumption that the observations at the sensors are independent, the output SINR is equal to the sum of the individual SINRs, and this is consequently an upper bound on performance. As a performance measure, we introduce the following metric:

$$\Gamma = \frac{\mathcal{E}^2\{\gamma_{\rm o}\}}{\text{var}\{\gamma_{\rm o}\}}. \tag{20}$$

The numerator is a measure of the average 'power' available in the system; the denominator is the variance, which is a measure of the uncertainty in the system. The overall metric is, therefore, a measure of the relative reliability of the system

As in the previous section, for tractability, we analyze the case of a single strong jammer. In this case, the metric for the overall system reduces to:

$$\Gamma = K\frac{\mathcal{E}^2\{\gamma_{\rm o}\}}{\text{var}\{\gamma_{\rm o}\}} = \frac{3KN(N-1)}{(N+1)(3N-2)}. \tag{21}$$

This expression allows one to investigate the tradeoff between K and N, i.e., of setting up a large number of smaller sensors versus having a small number of large sensors. To this end, let $\eta = KN$ be fixed. The metric of Eqn. (21) reduces to:

$$\Gamma = \frac{3K\frac{\eta}{K}(\frac{\eta}{K}-1)}{(\frac{\eta}{K}+1)(\frac{3\eta}{K}-2)} = 3\eta\frac{K(\eta-K)}{(\eta+K)(3\eta-2K)}. \tag{22}$$

We first prove that Γ increases with increasing K. Assuming, $N \geq 2$ such that the single jammer can be cancelled, hence $K \leq \eta/2$. Differentiating Γ with respect to K, we get:

$$\frac{d\Gamma}{dK} = \underbrace{\frac{3\eta^2}{(\eta+K)^2(3\eta-2K)^2}}_{\text{term 1}} \times \underbrace{(K^2-6K\eta+3\eta^2)}_{\text{term 2}}. \tag{23}$$

Term 1 is always positive. We now show that for feasible values of K, the second term is also positive. The roots of this polynomial are: $k_1 = \eta(3+\sqrt{6})$ and $k_2 = \eta(3-\sqrt{6}) \approx 0.55\eta$. Knowing that the above polynomial describes a convex parabola and that $K \leq \eta/2$ which is in turn smaller than the smaller root k_2, this means that the polynomial evaluates to positive values for all feasible K, hence $d\Gamma/dK$ is positive and Γ is strictly increasing with K.

This result shows that *as long as the interference is cancelled*, increasing the number of distributed sensors improves performance.

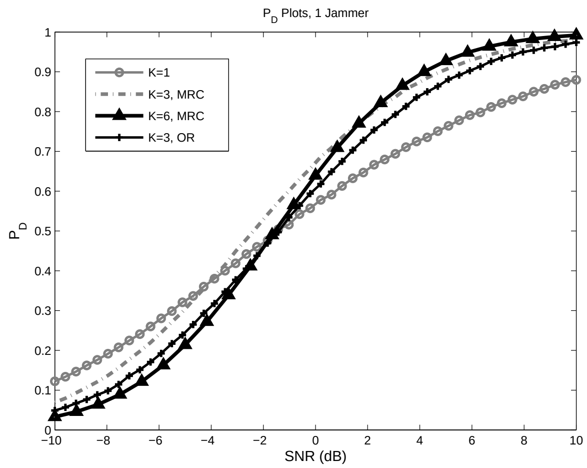
B. Probability of Detection

In what follows, we analyze the probability of detection for joint detection or maximal ratio combining (MRC) and the OR rule. Under both hypotheses, the test statistic (3) is exponentially distributed. For the OR rule, the total probability of detection is

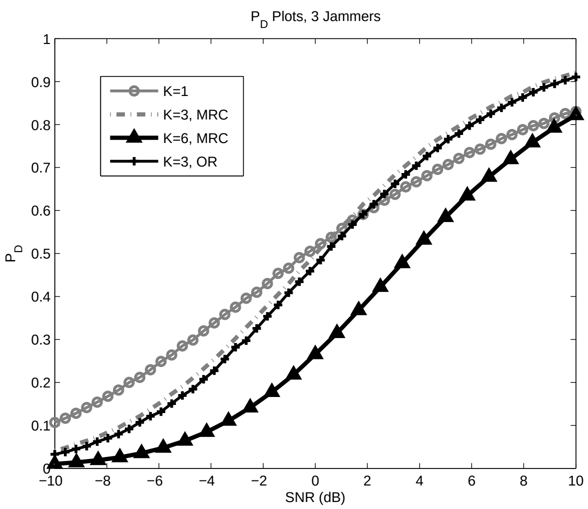
$$P_D = 1 - \prod_{k=1}^{K}(1-P_d^{(k)}), \tag{24}$$

where $P_d^{(k)}$ is the individual probability of detection at sensor k. In joint detection, each sensor contributes an exponentially distributed term with mean $\lambda_0 = A^2\mathbf{s}_k^H\mathbf{R}_n^{-1}\mathbf{s}_k/(1+A^2\mathbf{s}_k^H\mathbf{R}_n^{-1}\mathbf{s}_k)$ under H_0 and $\lambda_1 = A^2\mathbf{s}_k^H\mathbf{R}_n^{-1}\mathbf{s}_k$ under H_1. The statistic is a linear sum of exponential random variables and it follows a distribution of the form [4]:

$$f_{\sum x_j}(\zeta) = \left(\prod_{i=1}^{n}\lambda_i\right)\sum_{j=1}^{n}\frac{e^{-\lambda_j\zeta}}{\prod_{k=1,k\neq j}^{n}(\lambda_k-\lambda_j)}, \quad \zeta > 0. \tag{25}$$



(a) Average P_D plots with 1 jammer



(b) Average P_D plots with 3 jammers

Fig. 3. Plots for the average P_D with $J = 1, 3$.

This distribution is used to derive the probability of detection: first we invert this expression with the corresponding $\lambda_{i,0}$ to get the global threshold, we then use this threshold into the same expression, but now with $\lambda_{i,1}$ instead of $\lambda_{i,0}$ in order to calculate the probability of detection.

Figure 3 shows the average P_D results for a system with a total of $NK = 24$ antenna elements. Both MRC and the fully distributed OR rule are portrayed. Figure 3(a) shows the average P_D for 1 jammer. We first note that in this case, all the sensors are able to null the interference. For low input SNR, the larger sensors achieve better performance thanks to the array gain. When channel conditions improve, interference cancellation becomes the bottleneck to performance, and a larger number of sensors achieves better results. It is interesting to note that in this case the system exploits the spatial diversity as clearly manifested in the $K = 3$ and $K = 6$ cases. In terms of diversity order [1], the more sensors we have, the steeper the P_D curve is. This comes at the expense of poorer performance for low input SNR when the system is considered to be 'noise-limited'.

Figure 3(b) shows the average P_D for 3 jammers. The result is similar for $K = 3$ sensors. However, we see that for $K = 6$ which corresponds to $N = 4$, the sensors are not able to effectively null the 3 jammers, and the system achieves poor performance over all input SNR regimes. We note that the OR rule achieves results that are surprisingly close to optimum, especially when the interference environment become harsher.

V. Conclusions

The main contribution of this paper is the introduction of the notion of a random sensor network to help *analyze* distributed radar networks. Here we provide and analyze two theoretical models. For a unistatic system, where a single sensor is chosen to perform detection, we derive the SINR distribution, and provide expressions for the mean and the variance for a specific model. Multistatic detection is the main subject of this paper, where K sensors are randomly chosen to perform detection. We prove that in order to exploit the spatial diversity of the system, the sensors should be large enough to null the interference. We prove the existence of a fundamental tradeoff between interference cancelation and diversity: on one hand, larger sensors enhance the interference cancelation capability of the network. On the other hand, the existence of more sensors allows additional independent observations and hence larger diversity.

The work here and in [1] represent an initial attempt to develop a theory of distributed apertures used for detection. There has so far been little work on developing effective transmission and reception protocols coupled with effective signal processing algorithms that exploit the available diversity inherent in the system. In addition, an interesting theoretical extension to this work would be to formulate an optimization problem where power consumption is minimized given that the system achieves a certain diversity order and that a certain level of mobility is satisfied. Solving this problem will enable system designers to build efficient systems that both minimize power and maximize performance without sacrificing mobility.

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