

# Outage Probability at Arbitrary SNR in Cooperative Diversity Networks

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## Abstract

Cooperative diversity improves the performance of wireless networks by having several nodes transmit the same information. We present an outage probability analysis for a decode-and-forward system, valid at all signal to noise ratios (SNR). A closed form solution is obtained for independent and identically distributed (i.i.d.) channels, and two tight lower bounds are presented for correlated channels.

## I. INTRODUCTION

Cooperation helps create spatial diversity in wireless networks, even if individual nodes do not use antenna arrays for transmission or reception [1], [2]. While asymptotic (in SNR) performance analysis highlights the diversity order achievable by various techniques, it is also important to study performance in the non-asymptotic or finite SNR regime so as to compare various schemes in practical settings. In this letter, we derive the exact outage probability of the decode-and-forward scheme<sup>1</sup> for arbitrary signal-to-noise ratio (SNR) by using two powerful mathematical tools: Moment Generating Function (MGF) and Order Statistics. We derive an exact closed-form expression for i.i.d. channels, and introduce simple lower bounds for non-i.i.d. channels. Simulation results show that those bounds are very tight.

As an example of the utility of our results, consider that Laneman and Wornell proved in [1] that both the decode-and-forward and the space-time-coded cooperation can provide “full diversity” in the sense that the diversity order (using outage probability as a performance measure) is the total number of cooperating nodes in the network. However, our results will show that at finite SNR levels, a higher diversity order does not necessarily translate into better performance. This implies that the optimal number of cooperating nodes is in fact a complex function of the operating SNR and the cooperative diversity scheme in use.

## II. SYSTEM MODEL

We consider a system with a source node,  $s$ , communicating with a destination,  $d$ , with the help of  $m$  other cooperating (or relay) nodes, which are called cooperative nodes or relay nodes. Denote the set of all cooperating nodes as  $\{c\}$ . To guarantee orthogonal transmissions, we consider a Time Division Multiple

<sup>1</sup>Where the cooperative nodes attempt to decode the source’s bits, and retransmit those bits if the decoding succeeds [3],

Access (TDMA) arrangement with  $m + 1$  time slots. The first slot is used for the source to transmit its signal to the destination as well as share it with the cooperative nodes.

If the channel between the source and a node is good enough, this node becomes an *active* cooperating node, and decodes and forwards the source information. Denote the set of active cooperative nodes as  $C \in c$ . In the following  $m$  slots, the active cooperative nodes repeat the source message in a predetermined order [1].

Assuming that the destination  $d$  has exact channel state information (CSI), maximum-likelihood combining of the signals received from all  $|C| + 1$  nodes can be employed. Furthermore, we assume that both the source and the cooperative nodes do not have access to any transmit channel information.

### III. OUTAGE PROBABILITY: FORMULAS AND BOUNDS

#### A. Outage Probability for i.i.d. Channels

The mutual information between the source and cooperative nodes  $c = 1, \dots, m$  is [4]:

$$I_c = \frac{1}{m+1} \log(1 + \text{SNR}\hat{x}_c), \quad (1)$$

where  $\hat{x}_c = |h_{s,c}|^2$  and  $h_{s,c}$  is the complex channel gain between source  $s$  and node  $c$ , modelled as a zero-mean circularly symmetric Gaussian random variable.  $\hat{x}_c$  is exponentially distributed with parameter  $\hat{\lambda}_c$ . SNR is the transmit signal-to-noise ratio, and the factor  $\frac{1}{m+1}$  captures the TDMA nature of the scheme, in which node  $s$  is allowed to transmit its information only a fraction  $1/(m+1)$  of the time.

If the instantaneous mutual information  $I_c$  is higher than the transmission rate  $R$ , we can assume that the cooperative node successfully decodes the source bits, and thus belongs to the active cooperative set  $C$ , or  $C = \{c : I_c > R, c = 1, \dots, m\}$ .

As presented in [1], the mutual information of the decode-and-forward transmission is

$$I = \frac{1}{m+1} \log \left( 1 + \text{SNR} \left( x_0 + \sum_{c \in C} x_c \right) \right), \quad (2)$$

where  $x_0 = |h_{s,d}|^2$  and  $x_c = |h_{c,d}|^2$  are exponentially distributed with parameter  $\lambda_0$  and  $\lambda_c$ , respectively.

The outage probability is defined as  $P_{out} = P[I < R]$  where  $R$  is the required transmission rate for

source  $s$ . Using the total probability law, we can write the outage probability as:

$$P[I < R] = \sum_C P[I < R | C]P[C]. \quad (3)$$

Generally, this probability is very difficult to compute since the summation is over all the possible active cooperative sets, which has  $2^m$  items. However, if we assume independent and identically distributed (i.i.d.) fading, the closed-form outage probability can be derived.

*Theorem 1:* Under the assumption that the Rayleigh fading from the source to the cooperative nodes are i.i.d. ( $\hat{\lambda}_c = \hat{\lambda}$ ,  $c = 1, \dots, m$ ), and those from all nodes to the destination are also i.i.d. ( $\lambda_0 = \lambda_c = \lambda$ ,  $c = 1, \dots, m$ ), the outage probability of the system is

$$P_{out} = \sum_{k=0}^m \binom{m}{k} \left(e^{-\hat{\lambda}\gamma}\right)^k \left(1 - e^{-\hat{\lambda}\gamma}\right)^{m-k} \left[1 - e^{-\lambda\gamma} \sum_{i=0}^k \frac{(\lambda\gamma)^i}{i!}\right], \quad (4)$$

where  $\gamma = \frac{2^{(m+1)R}-1}{SNR}$ .

*Proof:* First consider the conditional probability

$$P[I < R | C = \{1, \dots, k\}] = P\left[\sum_{c=0}^k x_c < \frac{2^{(m+1)R} - 1}{SNR}\right] \quad (5)$$

We now use Moment Generating Function (MGF) to find the distribution of  $x_{sum} = \sum_{c=0}^k x_c$ . Since each  $x_c$  is exponentially distributed with parameter  $\lambda$ , its MGF is  $M_c(s) = \frac{\lambda}{s+\lambda}$ . Furthermore, the  $x_c$ 's are independent, so the MGF of  $x_{sum}$  is [5]

$$M_{sum}(s) = \left(\frac{\lambda}{s + \lambda}\right)^{k+1}. \quad (6)$$

Applying the inverse Laplace Transform, we can get the pdf, and then the CDF of  $x_{sum}$  as

$$F_{sum}(x) = 1 - e^{-\lambda x} \sum_{i=0}^k \frac{(\lambda x)^i}{i!}, \quad (7)$$

therefore (5) becomes

$$F_{sum}(\gamma) = 1 - e^{-\lambda\gamma} \sum_{i=0}^k \frac{(\lambda\gamma)^i}{i!}. \quad (8)$$

Notice that this conditional distribution is only determined by  $k$ , the size of the active set, and not by

the identity of the  $k$  nodes in the set. As a result, we can re-write (3) as

$$\mathbb{P}[I < R] = \sum_{k=0}^m \mathbb{P}[I < R \mid |C| = k] \mathbb{P}[|C| = k] \quad (9)$$

From the mutual information formula (1), we have

$$\mathbb{P}[c \in C] = \mathbb{P}[\hat{x}_c > \gamma] = e^{-\hat{\lambda}\gamma},$$

since  $\hat{x}_c$  is exponentially distributed with parameter  $\hat{\lambda}$ . It is straightforward then to arrive at

$$\mathbb{P}[|C| = k] = \binom{m}{k} \left(e^{-\hat{\lambda}\gamma}\right)^k \left(1 - e^{-\hat{\lambda}\gamma}\right)^{m-k} \quad (10)$$

Finally, substituting (8) and (10) into (9), we get (4), which completes the proof.  $\blacksquare$

### B. Convenient Lower Bounds

In Theorem 1 we derive the exact outage probability formula for i.i.d. fading. Unfortunately,  $P_{out}$  does not have a nice and simple form for general non-i.i.d. cases. However, we can still obtain convenient bounds to avoid the complex numerical summation in (3). We present two convenient lower bounds as the following theorems.

*Theorem 2:* One lower bound for the outage probability of a decode-and-forward system is

$$P_{out} \geq \left( \prod_{c=1}^m 1 - e^{-\hat{\lambda}_c \gamma} \right) (1 - e^{-\lambda_0 \gamma}) + \left( 1 - \prod_{c=1}^m 1 - e^{-\hat{\lambda}_c \gamma} \right) \left( \prod_{c=0}^m 1 - e^{-\lambda_c \gamma / (m+1)} \right). \quad (11)$$

*Proof:* Based on the fact that

$$x_0 + \sum_{c \in C} x_c \leq (m+1)x_{max}, \quad x_{max} = \max_{c=0, \dots, m} x_c,$$

first we have

$$P_{out} = \mathbb{P} \left[ x_0 + \sum_{c \in C} x_c < \gamma \right] \geq \mathbb{P}[x_0 < \gamma] \mathbb{P}[k = 0] + \mathbb{P}[(m+1)x_{max} < \gamma] \mathbb{P}[k \neq 0], \quad (12)$$

where  $k$  again represents the size of the active cooperative set, or  $k = |C|$ .

Using a result in order statistics, we can obtain the CDF of  $x_{max}$  as [6]

$$F_{max}(x) = \prod_{c=0}^m F_c(x) = \prod_{c=0}^m (1 - e^{-\lambda_c x}),$$

and therefore

$$P[(m+1)x_{max} < \gamma] = \prod_{c=0}^m (1 - e^{-\lambda_c \gamma / (m+1)}). \quad (13)$$

For the other items in (12), it is simple to obtain

$$P[k=0] = \prod_{c=1}^m (1 - e^{-\hat{\lambda}_c \gamma}) \quad (14)$$

$$P[k \neq 0] = 1 - \prod_{c=1}^m (1 - e^{-\hat{\lambda}_c \gamma}) \quad (15)$$

$$P[x_0 < th] = 1 - e^{-\lambda_0 \gamma}. \quad (16)$$

Substituting (13), (14), (15) and (16) into (12) completes the proof. ■

*Theorem 3:* Another lower bound for the outage probability of a decode-and-forward system is

$$P_{out} \geq \left( \prod_{c=1}^m 1 - e^{-\hat{\lambda}_c \gamma} \right) (1 - e^{-\lambda_0 \gamma}) + \left( 1 - \prod_{c=1}^m 1 - e^{-\hat{\lambda}_c \gamma} \right) \left( \sum_{c=0}^m a_c (1 - e^{-\lambda_c \gamma}) \right), \text{ where } a_c = \prod_{i \neq c} \frac{\lambda_i}{\lambda_i - \lambda_c}$$

*Proof:* Similar to (12), we have

$$P_{out} \geq P[x_0 < \gamma] P[k=0] + P\left[\sum_{c=0}^m x_c < \gamma\right] P[k \neq 0], \quad (17)$$

The MGF of  $\sum_{c=0}^m x_c$  can be written as

$$M_{sum}(s) = \prod_{c=0}^m \frac{\lambda_c}{s + \lambda_c} = \sum_{c=0}^m \frac{a_c \lambda_c}{s + \lambda_c}, \quad (18)$$

$$\text{where } a_c = \prod_{i \neq c} \frac{\lambda_i}{\lambda_i - \lambda_c}$$

Therefore the CDF of  $\sum_{c=0}^m x_c$  is

$$F_{sum}(x) = \sum_{c=0}^m a_c (1 - e^{-\lambda_c x}) \quad (19)$$

Finally, substituting (19), (14), (15) and (16) into (17) completes the proof. ■

Since  $x_0 + \sum_{c \in C} x_c \leq \sum_{c=0}^m x_c \leq (m+1)x_{max}$ , the second lower bound is tighter, but requires slightly

heavier computation load.

#### IV. SIMULATION RESULTS

Figure 1 shows the outage probability of a decode-and-forward system with the number of cooperative nodes ranging from 0 to 8. In the simulation we set  $R = 1$  bit/sec/Hz, and  $\hat{\lambda}_c = \lambda_0 = \lambda_c = 1, c = 1, \dots, m$ .

[Figure 1 about here.]

Plots like Figure 1 helps in system design not only because outage probability is an important QoS parameter in itself, but also because they allow us to determine the optimal number of cooperative nodes. For example, from Figure 1 we can see that when SNR= 20 dB, the optimal number of cooperative nodes is  $m = 2$ . When SNR increases to 30 dB, the optimal size changes to  $m = 4$ . An asymptotic analysis on the other hand will always point to  $m = 8$  as the best setting because it yields the largest diversity order.

Figure 2 compares the two lower bounds with the actual  $P_{out}$  obtained from numerical computations. In this simulation we set  $R = 1$  bit/sec/Hz,  $\hat{\lambda}_c, \lambda_0$  and  $\lambda_c$  are i.i.d. uniformly distributed in  $[0, 2]$ . From the figure we can see that the two bounds are almost identical and both are very tight, providing very good approximations to the outage probability.

[Figure 2 about here.]

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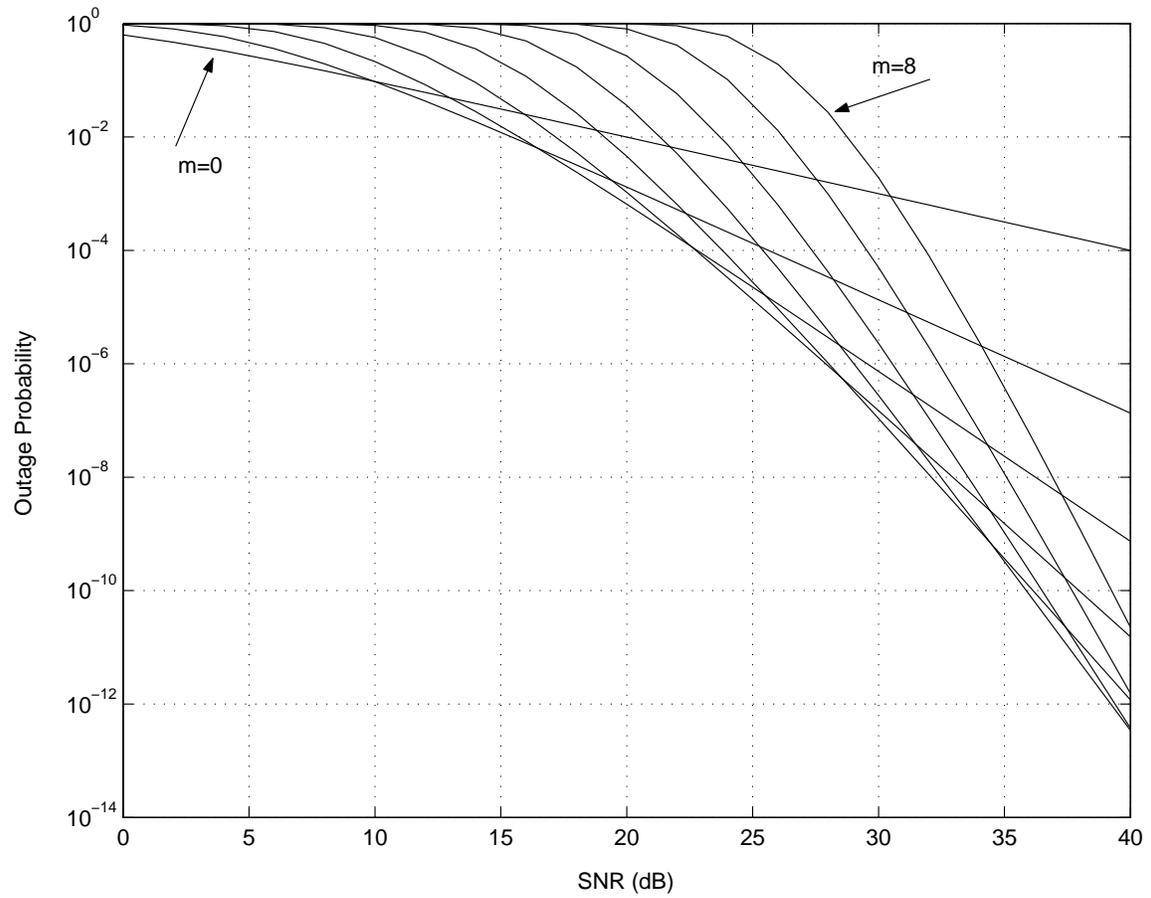


Fig. 1. Outage probability for the i.i.d. cases.

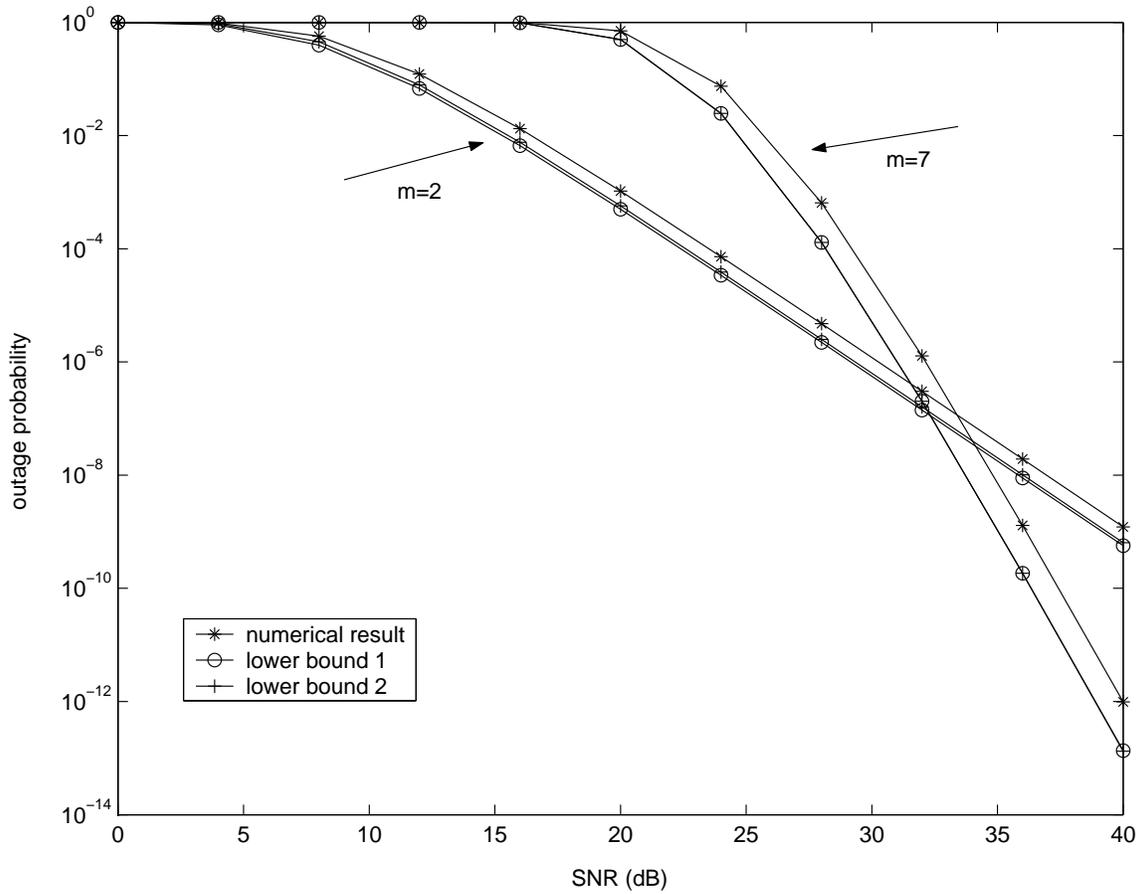


Fig. 2. Outage probability and lower bounds for the non-i.i.d. cases.