

# Linear Beamformer Design for Interference Alignment via Rank Minimization

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**Abstract**—This paper proposes a new framework for the design of transmit and receive beamformers for linear interference alignment (IA) without symbol extensions in MIMO cellular networks. We consider IA in a network consisting of  $G$  cells and  $K$  users/cell, having  $N$  and  $M$  antennas at each base station (BS) and user, respectively. In the proposed framework, the conditions for IA are shown to be equivalent to two sets of constraints, one governing the rank of *interference matrices*, and the other on the transmit beamformers in the uplink. The interference matrix consists of all the interfering vectors at a BS from the out-of-cell users. Using this set of conditions, this paper develops two sets of algorithms for IA. A crucial observation of this paper is that the rank of interference matrices under alignment can be identified based on the network parameters and the required degrees of freedom for all users, and that such prior knowledge can play an important role in the development of numerical algorithms for aligned beamformer design. The first part of this paper develops rank minimization algorithms that design aligned transmit beamformers in the uplink by iteratively minimizing a weighted matrix norm of the interference matrix. The choice between Frobenius norm and nuclear norm leads to reweighted nuclear norm minimization (RNNM) or reweighted Frobenius norm minimization (RFNM) with significantly different per-iteration complexities. By exploiting the prior knowledge of the required rank of interference matrices, we propose a novel weight update rule that navigates the algorithm towards aligned beamformers. As an alternative to solving rank minimization problems, the second part of this paper utilizes the prior rank knowledge in a different way and devises an alternating minimization (AM) algorithm where the rank-deficient interference matrices are expressed as a product of two lower-dimensional matrices. The two components are alternately optimized while keeping the other fixed. Simulation results indicate that RNNM, which has a per-iteration complexity of a semidefinite program, is effective in designing aligned beamformers for proper-feasible systems with or without redundant antennas, while RFNM and AM, which have a per-iteration complexity of a quadratic program, are better suited for systems with redundant antennas. All three algorithms are shown to outperform previous state-of-the-art algorithms.

## I. INTRODUCTION

Linear beamforming techniques for interference mitigation are of significant interest in MIMO cellular networks where

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increasing base-station (BS) density along with backhaul enhancements have necessitated and enabled coordinated interference management. In this context interference alignment has emerged as a key concept in addressing interference in such networks. In contrast to asymptotic interference alignment [1] that requires decomposition of multi-antenna nodes and infinite symbol extension, linear beamforming based alignment schemes are simpler to implement and therefore more relevant in practice. In this work we develop algorithms to design aligned transmit and receive beamformers to achieve a requisite number of degrees of freedom (DoF) in a  $G$ -cell,  $K$ -user/cell network with  $N$  antennas at each base station (BS) and  $M$  antennas at each user—a  $(G, K, N \times M)$  network, assuming channels to be generic with no symbol extensions.

Our interest in designing algorithms for interference alignment is twofold. First, in cooperative cellular networks that operate in the interference-limited regime, these beamformers identify regions in the optimization landscape where interference is significantly mitigated. Second, while techniques from algebraic geometry can be used to establish feasibility of interference alignment [2]–[5], techniques for the actual design of aligned beamformers still rely on numerical approaches. Although non-iterative approaches based on notions such as the subspace alignment chains for the 3-user interference channel [6] or the unstructured approach [7] for any  $G$ -cell,  $K$ -user network but under certain restrictions on  $N$ ,  $M$ , and the number of DoF required per user, iterative algorithms [8]–[16] continue to play an important role for general network parameters without any restrictions.

In this paper we develop two novel sets of iterative algorithms to design aligned beamformers. A crucial insight of this paper is that exploiting the prior knowledge of the rank of *interference matrices* can significantly benefit the numerical convergence of the algorithms. The interference matrix consists of all the interfering vectors received at a BS from the out-of-cell users. In the first set of algorithms, we use the prior rank knowledge to develop an effective algorithm to solve a rank minimization formulation. In an alternate approach, we restrict the rank of interference matrices by expressing them in a product form and develop a computationally efficient algorithm to find aligned beamformers.

### A. Literature Survey

Several iterative algorithms are available to design beamformers for interference alignment [8]–[18]. In [8], an iterative algorithm called iterative leakage minimization (ILM) is proposed for the MIMO interference channel. It is based on minimizing the sum of interference powers at all the receivers.

This algorithm is extended to MIMO cellular networks in [10]. While algorithms of [8] and [10] are known to converge to a locally optimal solution for minimizing interference power, empirically it is often the case that they need several hundred iterations to converge. However, their per-iteration complexity is low as they only require computing a small number of eigenvalue decompositions per iteration. More recently, [13], [14] also propose algorithms that minimize the total interference power across all the receivers. In [13], the total interference power is minimized while imposing a reduced row echelon form on the transmit and receive beamformers to ensure that they are full rank. In [14] a message passing framework to minimize total interference power is established and ILM is interpreted to be a special case of this framework.

In [11], a rank-constrained rank minimization (RCRM) framework for finding linear beamformers for interference alignment is proposed. Recognizing that the algorithms proposed in [8], [10] are equivalent to minimizing the Frobenius norm (matrix analog of the  $\ell_2$ -norm for vectors) of a projection of the interference matrix, the authors instead propose to minimize its nuclear norm. The projected interference matrix is the projection of the received interference vectors at a BS on the useful signal space as determined by the receive beamforming vectors at each BS. Nuclear norm minimization (the matrix analog of the  $\ell_1$ -norm for vectors) is known to induce sparsity and is well suited for generating aligned solutions. This framework is also capable of designing aligned beamformers when time or frequency extensions are allowed. Using the same framework, [15] and [16] further propose a reweighted version of nuclear norm minimization for finding aligned beamformers. Due to the nuclear norm approximation, the algorithms of [11], [15], [16] involve solving a series of semidefinite programs (SDPs). These algorithms are computationally more intensive, on a per-iteration basis, than the algorithms of [8], [10], but they require fewer iterations. However, in spite of using approximations which are sparsity inducing, these algorithms still lack explicit mechanisms to generate the desired level of sparsity for interference alignment and often fall short of generating the requisite number of interference free dimensions in practice—a shortcoming this paper seeks to address.

A key difference between [11], [15], [16] and this paper is that this paper works directly with the interference matrix rather than its projection using the receive beamformers. Note that the rank of the projected interference matrix when using aligned beamformers is zero. Clearly, this places an extreme sparsity requirement on the above approaches. As seen in the later sections of this paper, working directly with the interference matrices allows us to employ novel mechanisms to control the desired level of sparsity of the interference matrices.

Different from the above approaches, [17] and [18] develop algorithms to create sufficient number of interference-free dimensions for signal-vectors by minimizing a subset of eigenvalues of the interference matrix subject to a unitary constraint on the uplink transmit beamformers. The unitary constraint requires the optimization to be carried out on the complex Steifel manifold.

## B. Proposed Framework

In this work, we focus on linear interference alignment for generic channels without symbol extensions and reformulate the conditions for interference alignment as two sets of rank constraints, the first imposed on the interference matrix of each BS and the second on the transmit beamforming matrices. This framework is inspired by counting arguments that partition the total number of dimensions at each BS as either being occupied by signal or interference. The algorithms in [17], [18], developed originally for the MIMO interference channel, represent one particular approach to designing algorithms for interference alignment using such a framework.

Using this framework we develop three distinct and novel algorithms that aim to design uplink transmit beamformers for interference alignment. The first two algorithms, namely, the reweighted Frobenius norm minimization (RFNM) and reweighted nuclear norm minimization (RNNM) are developed by recasting the rank constraints as a rank minimization problem subject to linear constraints, while the third algorithm, called alternating minimization (AM), is developed by expressing rank-deficient matrices that satisfy the rank constraints as a product of two lower-dimensional matrices, then solving a convex quadratic program. In each of these cases, this paper makes a key observation that exploiting prior knowledge of the rank of interference matrices is crucial and that such an information can be inferred based on the network parameters and the required DoF for all users, as long as these required DoF/user are known to be achievable.

Establishing the achievability of certain DoF/user in a given  $(G, K, N \times M)$  network using linear beamforming without symbol extensions is actually a nontrivial problem. Efforts to establish achievability have lead to techniques based on algebraic geometry that provide an answer to feasibility without explicitly designing the beamformers [2]–[5]. The primary focus of this paper is to design aligned beamformers after the question of feasibility has been resolved. Thus, while we significantly rely upon previous works that aim to ensure feasibility, we also go beyond these papers and provide effective mechanisms to design the aligned beamformers.

1) *Reweighted Matrix Norm Minimization*: The rank minimization formulation aims to minimize the total number of dimensions occupied by interference at each BS by minimizing the rank of the interference matrix. Minimizing the dimensions occupied by interference opens up more dimensions for signal vectors. While the formulation in this paper is similar in spirit to [11], [17], [18], there are also significant differences. First, unlike [11], we only optimize over uplink transmit beamformers and eliminate the need to alternately optimize over transmit and receive beamformers. Second, instead of imposing rank constraints or unitary constraints on the uplink transmit beamformers, we impose a simple set of affine constraints that are straightforward to satisfy. These changes allow us to pose the problem of designing beamformers for interference alignment as a rank minimization problem subject to affine constraints only.

Rank minimization subject to affine constraints is well studied in the context of compressive sensing and there exist

several effective algorithms to solve such problems [19]–[25]. Drawing inspiration from these algorithms, we solve the rank minimization problem for interference alignment by approximating rank using a series of weighted Frobenius or nuclear norms. The iterative reweighting of the matrix norms provides improved approximation to the rank of the interference matrix after each iteration.

A crucial observation made in this paper is that utilizing prior rank knowledge of the interference matrix can significantly improve the convergence of the algorithms. Towards this end, we propose a novel reweighting rule that couples the weights in the weighting matrices by taking advantage of the prior knowledge of the expected rank of interference matrices. This coupling of weights acts as a control mechanism to indicate to the algorithm the desired level of sparsity, i.e., the desired ranks of the interference matrices.

The choice between reweighting the Frobenius and the nuclear norm results in two computationally different algorithms: RFNM, which minimizes a Frobenius norm, requires solving an unconstrained convex quadratic program at every iteration, and RNNM, which minimizes a nuclear norm, requires solving a semidefinite program. Although the per-iteration complexity of RNNM is significantly higher than that of RFNM, RNNM requires much fewer iterations to converge. This raises an interesting trade-off between complexity-per-iteration and the total number of iterations required.

2) *Alternating Minimization*: Recognizing rank deficiency of the interference matrices as the crucial bottleneck in designing beamformers for interference alignment, this paper further devises an alternative to rank minimization algorithms by explicitly imposing the rank constraint on the interference matrices. Since the desired ranks of the interference matrices are known *a priori*, we can express these rank-deficient matrices as a product of two lower-dimensional matrices. Once expressed in this manner, we can alternately optimize one of the two component matrices while holding the other constant. Such an approach, called alternating minimization (AM), has been previously proposed in the context of robust rank minimization in the presence of noisy observations [26], [27]. The optimization procedure can be interpreted as iterative minimization of the difference between the actual interference matrix as determined by the set of uplink transmit beamformers and a nominal interference matrix that identifies the subspace reserved for interference.

### C. Summary of Contributions

The main contributions in this paper include a new framework for interference alignment using which we develop: (a) Two new algorithms for interference alignment based on rank minimization subject to affine constraints; (b) An algorithm that uses alternating minimization to design aligned transmit beamformers in the uplink that satisfies the necessary rank constraints by expressing each interference matrix as a product of two lower-dimensional matrices. In both cases, exploiting the prior knowledge of the rank plays an important role in the effectiveness of the algorithms.

Key insight on system design from simulation results includes the following: (a) having redundant antennas in the

network significantly reduces the complexity of computing aligned beamformers — algorithms with low per-iteration complexity such as RFNM, AM and ILM are very effective in this case; (b) systems with no redundant antennas are the more challenging systems to design aligned beamformers for — the only effective algorithm for designing aligned beamformers for such systems is RNNM, where per-iteration complexity is that of an SDP.

### D. Paper Organization

This paper is organized as follows. In Sections II and III we introduce the system model and present a reformulation of the conditions for interference alignment. In Section IV, using this reformulation, we pose the problem of designing aligned beamformers as a rank minimization problem and develop the RFNM and RNNM algorithms. In Section V we develop the AM algorithm. Section VI compares the proposed algorithms with existing algorithms and Section VII presents the simulation results.

### E. Notation

We represent all column vectors in bold lower-case letters and all matrices in bold upper-case letters. The conjugate transpose and Euclidean norm of vector  $\mathbf{v}$  are denoted as  $\mathbf{v}^H$  and  $\|\mathbf{v}\|$ , respectively. The column span of the columns of a matrix  $\mathbf{X}$  is denoted as  $\text{span}(\mathbf{X})$ . The Frobenius norm and nuclear norm are denoted as  $\|\cdot\|_F$  and  $\|\cdot\|_*$  respectively. We use the simplified notation  $\{\mathbf{X}_{ij}\}$  to represent a set of matrices  $\mathbf{X}_{ij}$  where the indices  $i$  and  $j$  are within a range which is typically clear from the context. Similarly, we use the notation  $\{\mathbf{X}_{ij}\}_{j=1}^{j_{max}}$  when the index  $i$  is held constant and  $j$  is varied from 1 to  $j_{max}$ . We use the notation  $\mathbf{X}(1:i, 1:j)$  to refer to the submatrix of  $\mathbf{X}$  consisting of elements from the first  $i$  rows and first  $j$  columns of  $\mathbf{X}$ . The set of positive integers and the set of complex numbers are represented as  $\mathbb{Z}^+$  and  $\mathbb{C}$ , respectively.

## II. SYSTEM MODEL

Consider the uplink of a cellular network consisting of  $G$  interfering cells with  $K$  users per cell. Each user is assumed to have  $M$  antennas and each BS is assumed to have  $N$  antennas. Let the channel from the  $j$ th user in the  $i$ th cell to the  $g$ th base station (BS) be denoted as the  $N \times M$  matrix  $\mathbf{H}_{(i,j,g)}$ . We assume all channels to be generic. Equivalently, the channels can be assumed to be randomly drawn according to a cumulative distribution function in  $\mathbb{C}^{G^2 K M N}$  that is absolutely continuous and is such that all the conditional cumulative distribution functions of this distribution are also absolutely continuous. All channels are assumed to be known perfectly at a central location. The  $M \times 1$  signal vector transmitted by the  $k$ th user in the  $g$ th cell is formed using a  $M \times d_{gk}$  linear transmit beamforming matrix  $\mathbf{V}_{gk}$  and received using a  $N \times d_{gk}$  receive beamforming matrix  $\mathbf{U}_{gk}$ , where  $d_{gk}$  represents the number of transmitted data streams of the  $k$ th user in the  $g$ th cell. The received signal after being processed

by the receive beamforming matrix  $\mathbf{U}_{gk}$  at the  $g$ th BS can be written as

$$\mathbf{U}_{gk}^H \mathbf{y}_g = \sum_{i=1}^G \sum_{j=1}^K \mathbf{U}_{gk}^H \mathbf{H}_{(ij,g)} \mathbf{V}_{ij} \mathbf{s}_{ij} + \mathbf{U}_{gk}^H \mathbf{n}_g, \quad (1)$$

where  $\mathbf{s}_{ij}$  is the  $d_{ij} \times 1$  symbol vector transmitted by the  $j$ th user in  $i$ th cell and  $\mathbf{n}_g$  is the  $N \times 1$  vector representing circular symmetric additive white Gaussian noise  $\sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ . While the framework developed in this paper is applicable for any set  $\{d_{gk} : 1 \leq g \leq G, 1 \leq k \leq K, d_{gk} \in \mathbb{Z}^+\}$  that constitutes a proper and feasible system, we restrict our focus to the symmetric case where  $d_{gk} = d \forall g, k$ . The downlink received signal is defined similarly.

Note that although the development of the proposed algorithms is presented using the uplink signal model, the aligned beamformers that are eventually designed can be used in either uplink or downlink because of the duality of interference alignment [8].

We denote the space occupied by interference at the  $g$ th BS as the column span of a  $N \times (G-1)Kd$  matrix  $\mathbf{R}_g$  formed using the column vectors from the set  $\{\mathbf{H}_{(ij,g)} \mathbf{v}_{ijl} : i \in \{1, 2, \dots, G\}, j \in \{1, 2, \dots, K\}, l \in \{1, 2, \dots, d\}, i \neq g\}$ , where we use the notation  $\mathbf{v}_{ijl}$  to denote the  $l$ th beamformer associated with the  $j$ th user in the  $i$ th cell. For example, for the  $(3, 2, 4 \times 3)$  network with 1 DoF/user,  $\mathbf{R}_1$  is a  $4 \times 4$  matrix given by

$$\mathbf{R}_1 = [\mathbf{H}_{(21,1)} \mathbf{v}_{21} \quad \mathbf{H}_{(22,1)} \mathbf{v}_{22} \quad \mathbf{H}_{(31,1)} \mathbf{v}_{31} \quad \mathbf{H}_{(32,1)} \mathbf{v}_{32}]. \quad (2)$$

To enable the ability to measure and collect channel state information at a centralized location, this paper assumes the presence of backhaul links connecting all the BSs. These backhaul links allow the aligned beamformers to be communicated to each of these BSs. The BSs can use these beamformers in the downlink, over the air, so that the users can then locally compute their receive beamformers to null all the interference, thus achieving alignment.

### III. PROBLEM FORMULATION

The conditions for linear interference alignment when symbol extensions over time or frequency are not allowed can be stated as follows [28]:

$$\mathbf{U}_{gk}^H \mathbf{H}_{ij,g} \mathbf{V}_{ij} = \mathbf{0} \quad \forall (i, j) \neq (g, k) \quad (3)$$

$$\text{rank}(\mathbf{U}_{gk}^H \mathbf{H}_{gk,g} \mathbf{V}_{gk}) = d \quad \forall (g, k). \quad (4)$$

Counting the number of equations and variables involved in these conditions gives a preliminary check on the feasibility of satisfying (3) and (4). When the number of variables exceed the number of equations, the system is said to be proper. A  $(G, K, N \times M)$  network where  $d$  DoF/user are required is proper if  $(M + N) \geq (GK + 1)d$  [10], [28]. While not all proper systems are feasible [6], improper systems have been shown to be almost surely infeasible [2]. As mentioned earlier, this paper only considers proper systems whose feasibility has been established either through the techniques in [?], [2]–[4], or by other means.

The conditions for interference alignment can be alternately stated as

$$\text{rank}(\mathbf{R}_g) \leq N - Kd \quad \forall g, \quad (5)$$

$$\text{rank}(\mathbf{V}_{gk}) = d \quad \forall g, k. \quad (6)$$

This reformulation follows from the fact that when interference spans no more than  $N - Kd$  dimensions, generic channels ensure that the intersection between useful signal subspace ( $\text{span}(\{\mathbf{H}_{(gk,g)} \mathbf{V}_{gk}\}_{k=1}^K)$ ) and interference subspace ( $\text{span}(\mathbf{R}_g)$ ) is almost surely zero dimensional. This allows us to replace (3) with a rank constraint on the interference subspace (5). In addition, since generic channel matrices are almost always full rank, condition (4) is satisfied as long as  $\text{rank}(\mathbf{V}_{gk}) = d$ . This is because direct channels do not play a role in (5) and (6), and if  $\text{rank}(\mathbf{V}_{gk}) = d$  then  $\text{rank}(\mathbf{H}_{(gk,g)} \mathbf{V}_{gk}) = d$  almost surely and further,  $\text{rank}([\mathbf{H}_{(g1,g)} \mathbf{V}_{g1}, \dots, \mathbf{H}_{(gK,g)} \mathbf{V}_{gK}]) = Kd$  as well. This allows intra-cell interference to be nulled completely without losing signal dimensions, thus ensuring the existence of a valid set of receive beamformers at each BS. We collectively refer to the set of  $\{\mathbf{V}_{gk}\}$  as  $\mathbf{V}$ .

Since we need to design transmit beamformers that satisfy conditions (5) and (6), it is natural to pose the problem of finding these beamformers as the following feasibility problem:

$$\begin{aligned} & \text{minimize} && 1 \\ & \text{subject to} && \text{rank}(\mathbf{R}_g) \leq N - Kd \quad \forall g, \\ & && \text{rank}(\mathbf{V}_{gk}) = d \quad \forall (g, k), \\ & && \mathcal{A}_g(\mathbf{V}) = \mathbf{R}_g \quad \forall g. \end{aligned} \quad (7)$$

where  $\mathcal{A}_g(\mathbf{V}) = (\mathbf{R}_g)$  captures the linear relationship between  $\mathbf{V}$  and  $\mathbf{R}_g$  as shown in (2). While the rank constraint on  $\mathbf{V}_{gk}$  is easily handled by restricting it to be in column-reduced echelon form, i.e.,  $\mathbf{V}_{gk}(1:d, 1:d) = \mathbf{I} \forall g, k$ , handling the rank constraint on  $\mathbf{R}_g$  is not straightforward. Optimization problems involving rank constraints are very difficult to solve, requiring algorithms that traverse the manifold of low-rank matrices, or exploiting the structure of the problem to recast it in a more amenable form [29]–[31]. For the problem under consideration, we exploit the prior knowledge of the rank of  $\mathbf{R}_g$  and re-frame the rank constraint into numerically more amenable forms.

Towards this end, we propose two contrasting formulations. Substituting the rank constraints on  $\mathbf{V}_{gk}$  with equivalent affine constraints leaves us with only set of rank constraints to satisfy. In the first formulation we transform the feasibility problem to a rank minimization problem by moving the rank constraints on  $\mathbf{R}_g$  to the objective function. Such a transformation leads to an optimization problem of the form

$$\begin{aligned} & \min_{\mathbf{V}} \max_{g \in \{1, 2, \dots, G\}} \text{rank}(\mathbf{R}_g) \\ & \text{subject to} && \mathbf{V}_{gk}(1:d, 1:d) = \mathbf{I} \quad \forall (g, k) \\ & && \mathcal{A}_g(\mathbf{V}) = \mathbf{R}_g \quad \forall g. \end{aligned} \quad (8)$$

This is a minimax optimization problem where we minimize the maximum rank of the matrices  $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_G$ . Since we assume the given system to be proper and feasible, the

global optimum of this optimization problem is no more than  $N - Kd$ . Further, any set of beamformers that achieves the objective rank of  $N - Kd$  constitutes a set of aligned solutions. As shown in the next section, such prior knowledge of rank allows us to devise effective algorithms to find aligned beamformers. This prior knowledge is rarely available in generic rank minimization problems but is available in the context of interference alignment based on the network parameters and the required DoF for all the users. The required rank information for achieving interference alignment can then be inferred. This prior knowledge proves vital in reducing just the right number of singular values of the interference matrices to zero.

In the second formulation we relax the affine constraint governing  $\mathbf{V}$  and  $\mathbf{R}_g$  and move this constraint to the objective function by imposing a quadratic penalty on their difference  $(\mathcal{A}_g(\mathbf{V}) - \mathbf{R}_g)$ . This allows us to treat  $\mathbf{R}_g$  as a free variable which is independent of  $\mathbf{V}$  and lets us handle the rank constraint on  $\mathbf{R}_g$  by expressing it as a product of two matrices  $\mathbf{P}_g$  and  $\mathbf{Q}_g$  where  $\mathbf{P}_g$  is a  $N \times (N - Kd)$  matrix and  $\mathbf{Q}_g$  is a  $(N - Kd) \times (G - 1)Kd$  matrix, so that the rank of  $\mathbf{R}_g$  is no more than  $N - Kd$ . This reformulation leads to an optimization problem of the form

$$\begin{aligned} \min_{\mathbf{V}, \mathbf{P}_g, \mathbf{Q}_g} \max_{g \in \{1, 2, \dots, G\}} \quad & \|\mathcal{A}_g(\mathbf{V}) - \mathbf{R}_g\|_F^2 \\ \text{subject to} \quad & \mathbf{R}_g = \mathbf{P}_g \mathbf{Q}_g \quad \forall g, \\ & \mathbf{V}_{gk}(1 : d, 1 : d) = \mathbf{I} \quad \forall (g, k). \end{aligned} \quad (9)$$

Since all matrices permit a singular value decomposition (SVD), there is no loss of generality in expressing  $\mathbf{R}_g$  as the product  $\mathbf{P}_g \mathbf{Q}_g$ . Since we assume the given system to be proper and feasible, the global optimum of this optimization problem occurs when the objective is reduced to zero. The set of beamformers obtained at the global optimum satisfy the conditions for interference alignment.

Note that the two formulations differ in the constraint set over which the matrix  $\mathbf{R}_g$  is optimized. While the second formulation restricts the optimization to a set of rank deficient matrices, no similar rank condition is imposed in the first. Since rank deficient matrices constitute a measure-zero set over the space of all matrices of a given dimension, restricting the search space to a set of rank deficient matrices significantly prunes the search space. However, this gain comes at the cost having to define  $\mathbf{R}_g$  as a product of two matrices, thereby losing linearity.

Due to the non-smooth, non-convex nature of the rank function (8), the non-linearity in (9), and the difficulty in dealing with minimax formulation in both cases, further approximations to these formulations are necessary before developing algorithms to find aligned beamformers. We treat these two formulations in further detail in the next two sections.

#### IV. INTERFERENCE ALIGNMENT AS RANK MINIMIZATION

This section focuses on the first formulation where interference alignment is cast as a rank minimization problem. We first propose to replace the minimax formulation (8) by a min-

sum formulation given by

$$\begin{aligned} \text{minimize} \quad & \sum_{g=1}^G \text{rank}(\mathbf{R}_g) \\ \text{subject to} \quad & \mathbf{V}_{gk}(1 : d, 1 : d) = \mathbf{I} \quad \forall (g, k) \\ & \mathcal{A}_g(\mathbf{V}) = \mathbf{R}_g \quad \forall g. \end{aligned} \quad (10)$$

Although techniques such as the subgradient algorithm can be used to solve the minimax formulation directly, such techniques are known to converge very slowly. Replacing the minimax optimization problem with a min-sum optimization problem significantly alters the problem landscape. However, this paper shows that by taking advantage of the prior knowledge of the rank of optimal  $\mathbf{R}_g$ , this min-sum formulation can be effectively used to design aligned beamformers. This addresses a key difficulty in dealing with the original minimax formulation (8). A similar approach is adopted for solving (9) later in the paper.

Next, in order to apply standard optimization techniques for rank minimization, rank is approximated using a surrogate function. Well-known surrogate functions for rank include the nuclear norm (convex envelope of rank), the Schatten- $p$  function [25], the  $\log(\det(\cdot))$  function and the  $-\text{tr}(\text{inv}(\cdot))$  function [23], [24]. The choice of surrogate function determines the per-iteration complexity of the resulting algorithms. Solving (10) using the  $\log(\det(\cdot))$  approximation leads to a sequence of SDPs each minimizing a weighted Nuclear norm [19], while the Schatten- $p$  function requires solving a series of unconstrained convex quadratic programs each minimizing a weighted Frobenius norm. Either approximation leads to solving a series of convex optimization problems of the form

$$\begin{aligned} \text{minimize} \quad & \sum_{g=1}^G f_g^s(\mathbf{R}_g) \\ \text{subject to} \quad & \mathbf{V}_{gk}(1 : d, 1 : d) = \mathbf{I} \quad \forall (g, k) \\ & \mathcal{A}_g(\mathbf{V}) = \mathbf{R}_g \quad \forall g. \end{aligned} \quad (11)$$

where  $f_g^s(\mathbf{R}_g)$  is a convex function of  $\mathbf{R}_g$  that is used to approximate its rank in the  $s$ th iteration. The crucial element of such an approach is the iterative update of  $f_g^s(\cdot)$ . While the iterative updates suggested in [19], [25] are effective for rank minimization problems, they do not always minimize the rank of  $\mathbf{R}_g$  to the required extent for interference alignment in the min-sum formulation. The main idea of this paper is to develop a new update rule that ensures that the rank of  $\mathbf{R}_g$  is minimized to the desired extent while factoring in the minimax nature of the original formulation. This new approach is discussed for weighted Frobenius norm and for weighted nuclear norm minimization separately below.

##### A. Rank Minimization Using Frobenius Norm

1) *RFNM Algorithm*: We begin by discussing the standard approach for matrix rank minimization via reweighted Frobenius norm minimization. Solving affine-constrained rank minimization by iteratively solving a series of quadratic programs that minimize a weighted Frobenius norm is first discussed in [32], where the rank of a matrix  $\mathbf{X}$  is approximated using the

Schatten- $p$  function given by  $\text{tr}(\mathbf{X}^H \mathbf{X} + \gamma \mathbf{I})^{p/2}$  for  $0 < p \leq 1$ . Noting that the derivative of the Schatten- $p$  function is given by  $p\mathbf{X}(\mathbf{X}^H \mathbf{X} + \gamma \mathbf{I})^{p/2-1}$ , it is shown that the KKT conditions of the resulting affine-constrained optimization problem can be satisfied by iteratively solving a set of weighted-least-squares problems. Mathematically, the affine constrained rank minimization problem:

$$\begin{aligned} & \text{minimize} \quad \text{rank}(\mathbf{X}) \\ & \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{b}, \end{aligned} \quad (12)$$

is solved by iteratively solving the following quadratic optimization problem:

$$\begin{aligned} & \text{minimize} \quad \text{tr}(\mathbf{W}_s \mathbf{X}^H \mathbf{X}) = \|\mathbf{X}(\mathbf{W}_s)^{1/2}\|_F^2 \\ & \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{b}, \end{aligned} \quad (13)$$

where the weights  $\mathbf{W}_s$  are positive semidefinite matrices and are updated using the rule  $\mathbf{W}_{s+1} = ((\mathbf{X}_s)^H (\mathbf{X}_s) + \gamma_{s+1} \mathbf{I})^{\frac{p}{2}-1}$  where the optimal  $\mathbf{X}$  obtained after the  $s$ th iteration is denoted as  $\mathbf{X}_s$  and  $\gamma_{s+1}$  is the regularization parameter used in updating the  $(s+1)$ th weight. Further, the same iterative procedure can also be applied when  $p = 0$ , where the weight update rule is justified by showing that it solves a fixed point equation emerging from the KKT conditions that result when rank of  $\mathbf{X}$  is approximated as  $\log(\det(\mathbf{X}^H \mathbf{X} + \gamma \mathbf{I}))$ .

When weights are updated using the update rule given above, the weighting matrices can be interpreted to weight the singular values of the matrix  $\mathbf{X}_s^H \mathbf{X}_s$ . To see this, let the singular value decomposition of  $\mathbf{X}_s$  be given by  $\mathbf{P}_s \mathbf{\Sigma}_s \mathbf{Q}_s^H$ , then  $\mathbf{W}_{(s+1)} = \mathbf{Q}_s (\mathbf{\Sigma}_s^{(p-2)} + \gamma_{(s+1)} \mathbf{I})^{-1} \mathbf{Q}_s^H$ . Thus when  $p = 0$ , the weighting matrix  $\mathbf{W}_{s+1}$  imposes a penalty that is inversely proportional to the square of the magnitude of each non-zero singular value of  $\mathbf{X}_s$ . Since small, non-zero singular values are heavily penalized, the iterative procedure is incentivized to reduce them to zero, thus reducing the rank of  $\mathbf{X}$ .

This paper adapts the reweighted Frobenius norm minimization approach outlined above to solve (10) with  $p = 0$ , but adopts a novel reweighting strategy. This approach requires iterative solving of an optimization problem of the form

$$\begin{aligned} & \underset{\mathbf{V}}{\text{minimize}} \quad \sum_{g=1}^G \|\mathbf{R}_g (\mathbf{W}_{g(s)}^{1/2})^H\|_F^2 \\ & \text{subject to} \quad \mathbf{V}_{gk}(1:d, 1:d) = \mathbf{I} \quad \forall (g, k) \\ & \quad \mathcal{A}_g(\mathbf{V}) = \mathbf{R}_g \quad \forall g. \end{aligned} \quad (14)$$

where  $\mathbf{R}_g$  is weighted using  $\mathbf{W}_{g(s)}$  in the  $s$ th iteration. Note that we implicitly assume that  $\mathbf{R}_g$  has more rows than columns, if not, we simply replace  $\mathbf{R}_g$  with  $\mathbf{R}_g^H$  in the above formulation and all the subsequent steps. This ensures that the number of singular values of  $\mathbf{W}_{g(s)}$  and  $\mathbf{R}_g$  are the same. It turns out that a direct application of the weighting procedure outlined above to (14), while inducing a low-rank  $\mathbf{R}_g$ , almost never generates the total requisite number of interference-free dimensions. For example, the formulation in (14) may lead to scenarios where we have more than the necessary number of interference free dimensions at one BS with insufficient interference-free dimensions at other BSs. These issues are

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**Algorithm 1** Reweighted Frobenius Norm Minimization
 

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- 1: Initialize  $\mathbf{W}_{i(1)} = \mathbf{I} \forall i$ , set  $\gamma = \gamma_1$ ,  $s = 1$ .
  - 2: **while**  $s \leq \text{iter}_{max}$  **do**
  - 3: Solve (14) using weights  $\mathbf{W}_{i(s)}$  and denote the optimal interference matrices as  $\mathbf{R}_{i\text{opt}}$ .
  - 4: Compute the reduced SVD of  $\mathbf{R}_{i\text{opt}}$ , and denote it as  $\mathbf{P}_{i(s)} \mathbf{\Sigma}_{i(s)} (\mathbf{Q}_{i(s)})^H$ .
  - 5: Set  $\sigma_{min}^2 = \min_{i,r} \sigma_{ir}^2$ , where  $\sigma_{ir}$  are the singular values of  $\mathbf{R}_{i(s)}$  arranged in descending order.
  - 6: Set  $\mathbf{D}_{i(s)} = \text{diag}([\sigma_{i1}^2, \dots, \sigma_{i(N-Kd)}^2, \underbrace{\sigma_{min}^2, \dots, \sigma_{min}^2}_{z \text{ times}}])$ .
  - 7: Set  $\gamma = \max(\sigma_{min}^2, 10^{-8})$ .
  - 8: Update  $\mathbf{W}_{i(s+1)} = (\mathbf{Q}_{i(s)} \mathbf{D}_{i(s)} (\mathbf{Q}_{i(s)})^H + \gamma \mathbf{I})^{-1}$ .
  - 9: Set  $s = s + 1$ .
  - 10: **end while**
- 

effectively addressed by a new reweighting technique that exploits the prior knowledge of the required rank of  $\mathbf{R}_g$ .

2) *Proposed Reweighting Technique*: Since we are looking for transmit beamformers that ensure  $\text{rank}(\mathbf{R}_g) \leq N - Kd \forall g$ , we require  $z = \min(Kd, (GKd - N))$  singular values of  $\mathbf{R}_g$  to be zero. To avoid local minima where rank of  $\mathbf{R}_g$  is not sufficiently minimized, we couple the penalties (weights) associated with each of the  $z$  smallest singular values of  $\mathbf{R}_1, \mathbf{R}_2, \dots, \mathbf{R}_G$ . Let  $\{\sigma_{gr} : r = 1, 2, \dots, \min((G-1)Kd, N)\}$  be the set of singular values of  $\mathbf{R}_g$  obtained after the  $s$ th iteration, ordered in the descending order i.e.,  $\sigma_{gr} \geq \sigma_{g(r+1)}$ . Further, let

$$\sigma_{min}^2 = \min_{g,r} \sigma_{gr}^2, \quad (15)$$

and define the diagonal matrix  $\mathbf{D}_{g(s)}$  as

$$\mathbf{D}_{g(s)} = \text{diag}([\sigma_{g1}^2, \sigma_{g2}^2, \dots, \sigma_{g(N-Kd)}^2, \underbrace{\sigma_{min}^2, \sigma_{min}^2, \dots, \sigma_{min}^2}_{z \text{ times}}]). \quad (16)$$

We set the weights for the  $(s+1)$ th iteration to be

$$\mathbf{W}_{g(s+1)} = (\mathbf{Q}_{g(s)} \mathbf{D}_{g(s)} (\mathbf{Q}_{g(s)})^H + \gamma_{s+1} \mathbf{I})^{-1}, \quad (17)$$

where  $\mathbf{Q}_{g(s)}$  is the matrix of right singular vectors obtained from an SVD of the optimal  $\mathbf{R}_g$  obtained after the  $s$ th iteration. Such an update equally penalizes each of the  $z$  smallest singular values of  $\mathbf{R}_g \forall g$ , thereby encouraging the algorithm to seek aligned beamformers where all  $z$  smallest singular values can be simultaneously set to zero. The proposed iterative procedure is summarized as Algorithm 1.

The parameter  $\gamma$  acts as a regularization constant that makes sure the weighting matrices are positive definite. It limits the penalty imposed on small non-zero singular values and is typically reduced with each iteration to prevent the algorithm from prematurely converging to local minima.

Once the transmit beamformers in the uplink are designed, the receive beamformers at the  $g$ th BS for recovering the data streams of the  $k$ th user can be chosen to be the left-singular vectors corresponding to the  $d$  smallest singular values of the

matrix  $\mathbf{R}_g$  augmented with the interfering vectors from other users in the same cell. That is, suppose  $\mathbf{P}\Sigma\mathbf{Q}^H$  represents the SVD of the matrix

$$[\mathbf{R}_g \mathbf{H}_{(g1,g)} \mathbf{V}_{g1} \cdots \mathbf{H}_{(g(k-1),g)} \mathbf{V}_{g(k-1)} \mathbf{H}_{(g(k+1),g)} \mathbf{V}_{g(k+1)} \cdots \mathbf{H}_{(gK,g)} \mathbf{V}_{gK}],$$

where  $\Sigma$  has singular values listed in the decreasing order, then we set

$$\mathbf{U}_{gk} = \mathbf{P}(1 : N, N - d + 1 : N). \quad (18)$$

Such a choice of uplink receive beamformers ensures that both inter-cell and intra-cell interference are minimized for the given set of uplink transmit beamformers. When the uplink transmit beamformers are the aligned beamformers, this choice of receive beamformers completely null out inter-cell and intra-cell interference, thus achieving alignment.

3) *Convergence and Complexity*: When weights are updated according to the original update rule given in [25] with  $p = 0$ , the value of the surrogate function can be shown to converge; a proof of convergence of the iterates is not yet known. For the proposed weight update rule convergence of the iterates and the surrogate function is not guaranteed. However, since decoupling the weights ensures convergence, we can decouple the weights after a fixed number of iterations to let the algorithm converge. In our simulations we simply run the algorithm for a fixed number of iterations ( $iter_{max}$ ) with coupled weights.

To analyze the complexity of the algorithm, note that each iteration of the proposed heuristic requires solving the quadratic program (14) and computing SVDs of  $G$  matrices of size  $N \times (G-1)Kd$ . The two linear constraints in (14) are easy to eliminate as  $\mathbf{R}_g$  is just an auxiliary variable and the conditions on  $\mathbf{V}_{gk}$  are easy to satisfy. Thus, (14) is an unconstrained quadratic program in  $GK(M-d)d$  (denote this as  $\nu$ ) variables and requires solving a linear system in the same number of variables which can be accomplished in  $O(\nu^3)$  time. Each SVD can be computed in  $O(\max(N, (G-1)Kd)^3)$  time.

## B. Rank Minimization Using Nuclear Norm

1) *RNNM Algorithm*: We now present an analogous reweighting strategy for solving the interference alignment problem by rank minimization where the rank is approximated by nuclear norm. Rank minimization is the matrix analog of  $\ell_0$ -norm minimization for vectors, and techniques developed for  $\ell_0$ -norm minimization can be extended to rank minimization. In the compressive sensing literature,  $\ell_0$ -norm minimization is often approximated by reweighted  $\ell_1$ -norm minimization [20]. The matrix counterpart of reweighted  $\ell_1$ -norm minimization proposed in [20] is reweighted nuclear norm minimization [19]. This procedure reformulates the rank minimization in (10) as

$$\begin{aligned} & \text{minimize } \sum_{g=1}^G \|\mathbf{W}_{gL}\mathbf{R}_g\mathbf{W}_{gR}\|_* \\ & \text{subject to } \mathbf{V}_{gk}(1:d, 1:d) = \mathbf{I} \quad \forall(g, k) \\ & \quad \mathbf{A}_g(\mathbf{V}) = (\mathbf{R}_g) \quad \forall g. \end{aligned} \quad (19)$$

where  $\mathbf{W}_{gL}$  and  $\mathbf{W}_{gR}$  are two positive definite matrices that are interpreted to reweight the nuclear norm of  $\mathbf{R}_g$ . The iterative procedure involves solving an instance of (19) for fixed  $\mathbf{W}_{gL}$  and  $\mathbf{W}_{gR}$ , then updating the weights for the next iteration. Every iteration requires solving a semidefinite program.

The choice of the weight update rule affects the overall performance and needs to be chosen carefully. In [19], the authors derive a weight update rule by exploiting an equivalence relation between the rank of  $\mathbf{R}_g$  and a positive semidefinite matrix  $\mathbf{Z}$  of the form  $\begin{bmatrix} \mathbf{A} & \mathbf{R}_g \\ \mathbf{R}_g^H & \mathbf{B} \end{bmatrix}$ , and approximating the rank of  $\mathbf{Z}$  using the  $\log(\det(\cdot))$  surrogate function. This weight update rule is interpreted to minimize this concave surrogate function through a majorization-minimization procedure [19], [20], [23]. The resulting weights can be interpreted as the reweighting of the singular values of the matrix  $\mathbf{Z}$ .

2) *Proposed Reweighting Technique*: While we adopt the general framework of reweighting as proposed in [19], we update weights using an update rule similar to that proposed in the previous section. Note that we require  $z = \min(Kd, (GKd-N))$  of the  $y = \min(N, (G-1)Kd)$  singular values of  $\mathbf{R}_g$  to be zero. Let  $\mathbf{P}_{gs}\Sigma_{gs}(\mathbf{Q}_{gs})^H$  be the full singular value decomposition of the optimum  $\mathbf{R}_g$  obtained after the  $s$ th iteration and let  $\sigma_{min} = \min_{g,k} \sigma_{gk}$ , where  $\sigma_{gk}$  are the singular values of  $\Sigma_{gs}$ . We once again couple the penalties associated with those singular values that need to be set to zero and update the weights for the  $(s+1)$ th iteration as follows

$$\mathbf{W}_{gL(s+1)} = \mathbf{P}_{gs}(\mathbf{D}_{gL(s+1)} + \gamma\mathbf{I})^{-\frac{1}{2}}(\mathbf{P}_{gs})^H \quad (20)$$

$$\mathbf{W}_{gR(s+1)} = \mathbf{Q}_{gs}(\mathbf{D}_{gR(s+1)} + \gamma\mathbf{I})^{-\frac{1}{2}}(\mathbf{Q}_{gs})^H \quad (21)$$

where  $\mathbf{D}_{gL(s+1)}$  and  $\mathbf{D}_{gR(s+1)}$  are defined as

$$\mathbf{D}_{gL(s+1)} = \text{diag}([\sigma_{g1}, \sigma_{g2}, \dots, \sigma_{g(y-z)}, \underbrace{\sigma_{min}, \sigma_{min}, \dots, \sigma_{min}}_{N-y+z \text{ times}}]) \quad (22)$$

$$\mathbf{D}_{gR(s+1)} = \text{diag}([\sigma_{g1}, \sigma_{g2}, \dots, \sigma_{g(y-z)}, \underbrace{\sigma_{min}, \sigma_{min}, \dots, \sigma_{min}}_{(G-1)Kd-y+z \text{ times}}]) \quad (23)$$

where the singular values are assumed to be ordered in the descending order. Again, the idea is to equally penalize the singular values that are to be reduced to zero so that  $\mathbf{R}_g$  is of the requisite rank. The proposed iterative procedure is summarized as Algorithm 2.

3) *Convergence and Complexity*: A key difference between RFNM and RNNM is that every iteration of RNNM requires solving an SDP. SDPs are typically solved using the primal-dual interior point method [33], as in popular solvers such as SDPT3 [34], [35]. The SDP in (19) requires minimizing the rank of a block diagonal matrix consisting of  $G$  blocks each of size  $N \times (G-1)Kd$ . This problem can be posed in the standard SDP form using Lemma 1 of [36]. Excluding auxiliary variables such as  $\mathbf{R}_g$ , when posed in standard form

**Algorithm 2** Reweighted Nuclear Norm Minimization

- 1: Initialize  $\mathbf{W}_{iL1} = \mathbf{I}$ ,  $\mathbf{W}_{iR1}^1 = \mathbf{I} \forall i$ ,  $\gamma = 10^{-6}$ ,  $s = 1$ .
- 2: **while**  $s \leq iter_{max}$  **do**
- 3: Solve (19) using weights  $\mathbf{W}_{iLs}$  and  $\mathbf{W}_{iRs}$ ; Denote the optimal interference matrices as  $\mathbf{R}_{is}$ .
- 4: Compute the full SVD of  $\mathbf{R}_{is}$ , and denote it as  $\mathbf{P}_{is}\mathbf{\Sigma}_{is}(\mathbf{Q}_{is})^H$ .
- 5: Set  $\sigma_{min} = \min_{i,j} \sigma_{ij}$ , where  $\sigma_{ij}$  are the singular values of  $\mathbf{R}_i$  arranged in descending order.
- 6: Set  $\mathbf{D}_{iLs}$  and  $\mathbf{D}_{iRs}$  as given in (22) and (23).
- 7: Compute  $\mathbf{W}_{iL(s+1)}$  and  $\mathbf{W}_{iR(s+1)}$  using (20) and (21).
- 8: Set  $s = s + 1$ .
- 9: **end while**

there are a total of  $\nu_{sdp}$  free variables, where

$$\nu_{sdp} = GKd(M-d) + N^2/2 + ((G-1)Kd)^2/2. \quad (24)$$

The per-iteration complexity of the primal-dual interior point method used in generic solvers when applied to RNNM grows at least as fast as  $O(\nu_{sdp}^3)$  [37]. Custom SDP solvers that exploit structure in the SDP can reduce the per-iteration complexity to  $O(\nu_{sdp-custom})$  [37], where  $\nu_{sdp-custom}$  is given by

$$\nu_{sdp-custom} = N \times (G-1)Kd \times (GK(M-d)d)^2. \quad (25)$$

In general the number of iterations of the primal-dual interior point method required to solve one instance of (19) is between 10-50 iterations and is only weakly dependent on the network parameters. After solving the SDP, further computations are necessary to compute the  $G$  SVDs to reweight the nuclear norm. Clearly, the complexity of solving one instance of (19) is considerably higher than that of (14). However, RNNM only requires solving about 20-40 instances of (19) as opposed to several hundred instances of (14) in the case of RFNM. It is hence difficult to compare the overall complexity of RFNM and RNNM.

Although a proof of convergence for RNNM using the proposed weight update rule is not yet available, simulation results indicate that the algorithm converges within tens of iterations. In our simulations we run the algorithm for a fixed number of iterations ( $iter_{max}$ ).

## V. INTERFERENCE ALIGNMENT USING AM

### A. AM Algorithm

We now turn to the second formulation (9) for the interference alignment problem. As seen in (9), relaxing the linear constraint between the interference matrix  $\mathbf{R}_g$  and the transmit beamformers  $\mathbf{V}$  by imposing a quadratic penalty on the difference allows us to treat  $\mathbf{R}_g$  as a free variable and to express it as the product  $\mathbf{P}_g\mathbf{Q}_g$ . This new approach to satisfying rank constraints on  $\mathbf{R}_g$  is inspired by the alternating minimization algorithm proposed in the context of robust rank minimization [26], [27]. Such an approach proposes to iteratively optimize either  $\mathbf{P}_g$  or  $\mathbf{Q}_g$  while holding the other

**Algorithm 3** Alternating Minimization

- 1: Initialize  $\mathbf{Q}_i$  for  $i \in \{1, 2, \dots, G\}$  with random entries drawn from a continuous distribution. Set  $s = 1$ .
- 2: **while**  $s \leq iter_{max}$  **do**
- 3: Solve the following optimization problem using  $\mathbf{Q}_i$  obtained from Step 4 (for the first iteration use  $\mathbf{Q}_i$  initialized to random entries):

$$\begin{aligned} & \underset{\mathbf{V}, \{\mathbf{P}_i\}}{\text{minimize}} && \sum_{g=1}^G \|\mathcal{A}_g(\mathbf{V}) - \mathbf{P}_g\mathbf{Q}_g\|_F^2 \\ & \text{subject to} && \mathbf{V}_{ij}(1:d, 1:d) = \mathbf{I} \forall (i, j). \end{aligned} \quad (27)$$

- 4: Solve the following optimization problem using  $\mathbf{P}_i$  obtained from Step 3.

$$\begin{aligned} & \underset{\mathbf{V}, \{\mathbf{Q}_i\}}{\text{minimize}} && \sum_{g=1}^G \|\mathcal{A}_g(\mathbf{V}) - \mathbf{P}_g\mathbf{Q}_g\|_F^2 \\ & \text{subject to} && \mathbf{V}_{ij}(1:d, 1:d) = \mathbf{I} \forall (i, j) \end{aligned} \quad (28)$$

- 5: Set  $s = s + 1$ .
- 6: **end while**

constant. With computational complexity in mind, we again simplify (9) to a min-sum formulation as given below:

$$\begin{aligned} & \underset{\mathbf{V}, \mathbf{P}_g, \mathbf{Q}_g}{\text{minimize}} && \sum_{g=1}^G \|\mathcal{A}_g(\mathbf{V}) - \mathbf{R}_g\|_F^2 \\ & \text{subject to} && \mathbf{R}_g = \mathbf{P}_g\mathbf{Q}_g \quad \forall g, \\ & && \mathbf{V}_{gk}(1:d, 1:d) = \mathbf{I} \quad \forall (g, k). \end{aligned} \quad (26)$$

Note that the global optimum is still zero and that all sets of aligned beamformers achieve the global optimum. When either  $\mathbf{P}_g$  or  $\mathbf{Q}_g$  is fixed, the above formulation is a convex quadratic program. We solve (26) by alternately solving for  $\mathbf{P}_g$  and  $\mathbf{Q}_g$  while keeping the other fixed. Intuitively,  $\mathbf{P}_g$  and  $\mathbf{Q}_g$  jointly define the space occupied by interference at the  $g$ th BS, and the beamformers are designed to make sure that the residual interference that spills beyond the space identified by  $\mathbf{P}_g$  and  $\mathbf{Q}_g$  is minimized at every iteration. The proposed algorithm is summarized in Algorithm 3.

### B. Convergence and Complexity

Since every step in alternating minimization decreases the quadratic objective, the value of the objective converges as it is bounded below by zero. Although the iterates ( $\mathbf{P}_g$ ,  $\mathbf{Q}_g$  and  $\mathbf{V}_{gk}$ ) themselves cannot be guaranteed to converge, this has no significant impact on the algorithm.

To analyze the complexity of the proposed algorithm, note that the equality constraints in (27) and (28) are trivial to satisfy. Solving (27) and (28) requires finding the minimum of an unconstrained convex quadratic function involving  $GKd(M-d) + GN(N-Kd)$  (denote this as  $\nu_p$ ) complex scalar variables in the case of (27) and  $GKd(M-d) + G(G-1)Kd(N-Kd)$  (denote this as  $\nu_q$ ) complex scalar variables in the case of (28). This requires solving a set of linear equations involving  $\nu_p$  and  $\nu_q$  variables respectively and the worst-case

complexity of solving this set of linear equations is given by  $O(\nu_p^3)$  and  $O(\nu_q^3)$  respectively. This algorithm is comparable in complexity to RFNM as it requires solving a simple quadratic program. In addition, it does not require computing any SVDs.

## VI. COMPARISON TO EXISTING ALGORITHMS

We provide a brief discussion on some of the existing algorithms for the purpose of performance comparison. Two well-known algorithms for interference alignment are ILM [8]–[10] and RCRM [11]. ILM aims to minimize the overall interference power across the network by solving the following optimization problem:

$$\begin{aligned} & \underset{\mathbf{V}_{gk}, \mathbf{U}_{gk}}{\text{minimize}} && \sum_{g=1}^G \|\mathbf{U}_{g1}, \dots, \mathbf{U}_{gK}\|^2_{\mathbf{R}_g} \\ & \text{subject to} && \text{rank}(\mathbf{V}_{gk}) = d \quad \forall (g, k) \\ & && \text{rank}([\mathbf{U}_{g1}, \dots, \mathbf{U}_{gK}]) = Kd \quad \forall g \end{aligned} \quad (29)$$

Note that only inter-cell interference is taken into account as intra-cell interference can be nulled subsequently by an appropriate linear transformation of the receive beamformers ( $\{\mathbf{U}_{gk}\}$ ). The algorithm alternately optimizes the transmit and receive beamformers and involves computing  $G(K+1)$  eigenvalue decompositions in each iteration.

Interpreting interference alignment as reducing the rank of the projected interference matrix, RCRM [11] replaces Frobenius norm with nuclear norm and considers solving the following optimization problem

$$\begin{aligned} & \underset{\mathbf{V}_{gk}, \mathbf{U}_{gk}}{\text{minimize}} && \sum_{g=1}^G \|\mathbf{U}_{g1}, \dots, \mathbf{U}_{gK}\|_{\mathbf{R}_g} \\ & \text{subject to} && \lambda_{\min}(\mathbf{U}_{gk}^H \mathbf{H}_{(gk,g)} \mathbf{V}_{gk}) \geq \epsilon \quad \forall (g, k) \end{aligned} \quad (30)$$

where  $\epsilon > 0$ . The constraint on the minimum eigenvalue of the received signal space ensures that: (a) all transmit and receive beamformer matrices have rank  $d$ , and (b) the received signal space  $\mathbf{U}_{gk}^H \mathbf{H}_{(gk,g)} \mathbf{V}_{gk}$  is not rank deficient. For any choice of  $\epsilon > 0$ , the constraint on the smallest eigenvalue places a restriction on the received signal space and the optimization is technically not a pure pursuit to align interference. While this is a necessary constraint when channels are not generic, this constraint can be dropped for generic channels provided  $\mathbf{U}_{gk}$  and  $\mathbf{V}_{gk}$  are all full rank. The optimization problem (30) is convex in either  $\{\mathbf{U}_{gk}\}$  or  $\{\mathbf{V}_{gk}\}$  when the other is held fixed and is solved by alternately optimizing  $\{\mathbf{U}_{gk}\}$  and  $\{\mathbf{V}_{gk}\}$ . Each iteration requires solving two SDPs, one each to update  $\{\mathbf{U}_{gk}\}$  and  $\{\mathbf{V}_{gk}\}$ .

In [16], instead of approximating rank using nuclear norm in the RCRM formulation, rank is approximated using the  $\log(\det(\cdot))$  surrogate function. A majorization-minimization routine that minimizes a series of weighted nuclear norms is proposed. The resulting algorithm consists of two loops—the outer loop updates the weights, while the inner loop alternately optimizes transmit and receive beamformers for a fixed set of weights. The core optimization problem solved at each step is

TABLE I: Network parameters used in simulations. In all systems, the number of data streams per user is  $d = 1$ .

	System ( $G, K, N \times M$ )	Redun- dancy	Total no. of data streams	Interference-free dimensions to be created per BS
1	(4, 1, 3 × 2)	No	4	1
	(4, 1, 3 × 3)	Yes	4	1
2	(3, 2, 4 × 3)	No	6	2
	(3, 2, 4 × 4)	Yes	6	2
3	(3, 3, 6 × 4)	No	9	3
	(3, 3, 7 × 4)	Yes	9	2

similar to (30) and is given by

$$\begin{aligned} & \underset{\mathbf{V}_{gk}, \mathbf{U}_{gk}}{\text{minimize}} && \sum_{g=1}^G \|\mathbf{W}_g[\mathbf{U}_{g1}, \dots, \mathbf{U}_{gK}]\|_{\mathbf{R}_g} \\ & \text{subject to} && \lambda_{\min}(\mathbf{U}_{gk}^H \mathbf{H}_{(gk,g)} \mathbf{V}_{gk}) \geq \epsilon \quad \forall (g, k). \end{aligned} \quad (31)$$

where  $\mathbf{W}_g$  is the weighting matrix. In the outer loop, the weighting matrices are updated as  $\mathbf{P}_g(\sum_g + \gamma \mathbf{I})^{-1} \mathbf{P}_g^H$ , where  $\gamma$  is a regularizing constant and  $\mathbf{P}_g \sum_g \mathbf{Q}_g^H$  is the SVD of the optimal  $[\mathbf{U}_{g1}, \dots, \mathbf{U}_{gK}]^H \mathbf{R}_g$  matrix obtained from the inner loop. Note that unlike the reweighting rule in Section IV-B, the weight update rule in [16] does not couple the weights. Further, since all the singular values of the interference matrices are to be reduced to zero, the idea of coupling weights does not provide any benefit as the formulation then becomes equivalent to RCRM.

The ILM framework introduced in [8] serves as an important starting point for the subsequent algorithms discussed above. The subsequent RCRM formulations are developed with the goal of inducing sparsity to reduce the dimensions occupied by interference (after being projected by receive beamformers). When interference alignment is feasible, the singular values of all the matrices in the objective of (30) and (31) are to be reduced to zero. However, this places an extreme sparsity requirement on the optimization problems as the rank of a  $GKd \times G(G-1)Kd$  block diagonal matrix needs to be reduced to zero. Typical rank minimization algorithms are not designed to generate this level of sparsity and quite often do not succeed in completely eliminating interference.

As compared to ILM and RCRM, the new algorithms proposed in this paper have several key advantages. First, by focusing on the received interference vectors before being projected by the receive beamformers, the algorithms developed in this paper do not require alternately optimizing transmit and receive beamformers. Further, we incorporate prior knowledge of the requisite number of interference-free dimensions in designing our algorithms. Such a design prevents the algorithms from converging to local minima and helps reduce the rank of the interference matrices to the required extent.

## VII. SIMULATION RESULTS

To evaluate the algorithms developed in this paper, we consider networks of varying sizes and number of antennas,

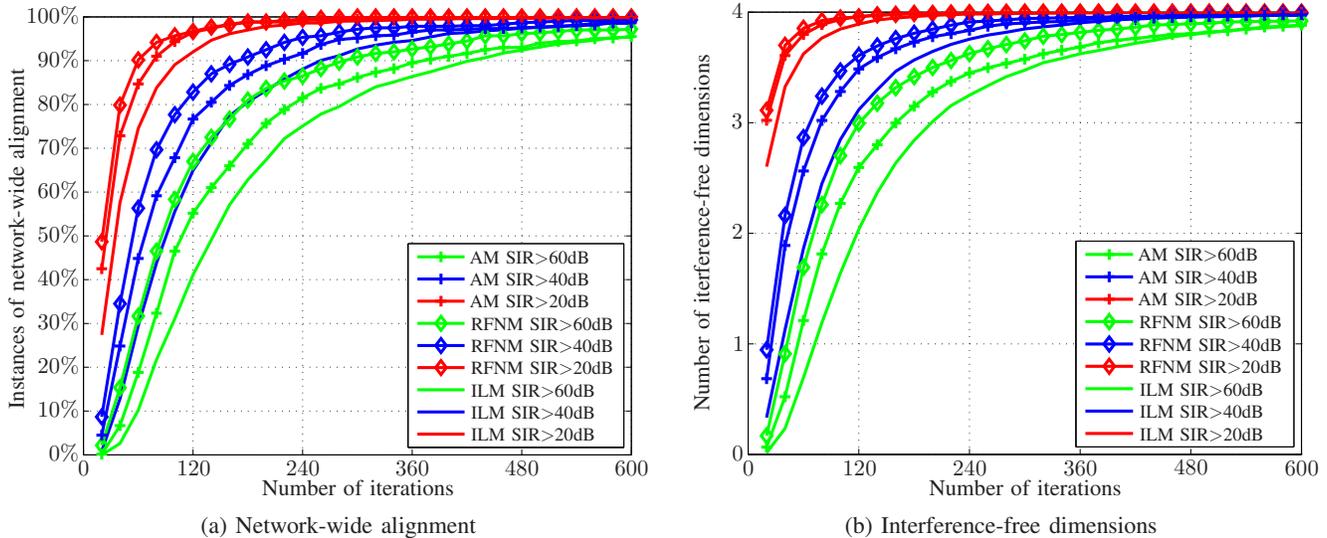


Fig. 1: Alignment in the  $(4, 1, 3 \times 2)$  network using the RFNM, AM and ILM algorithms.

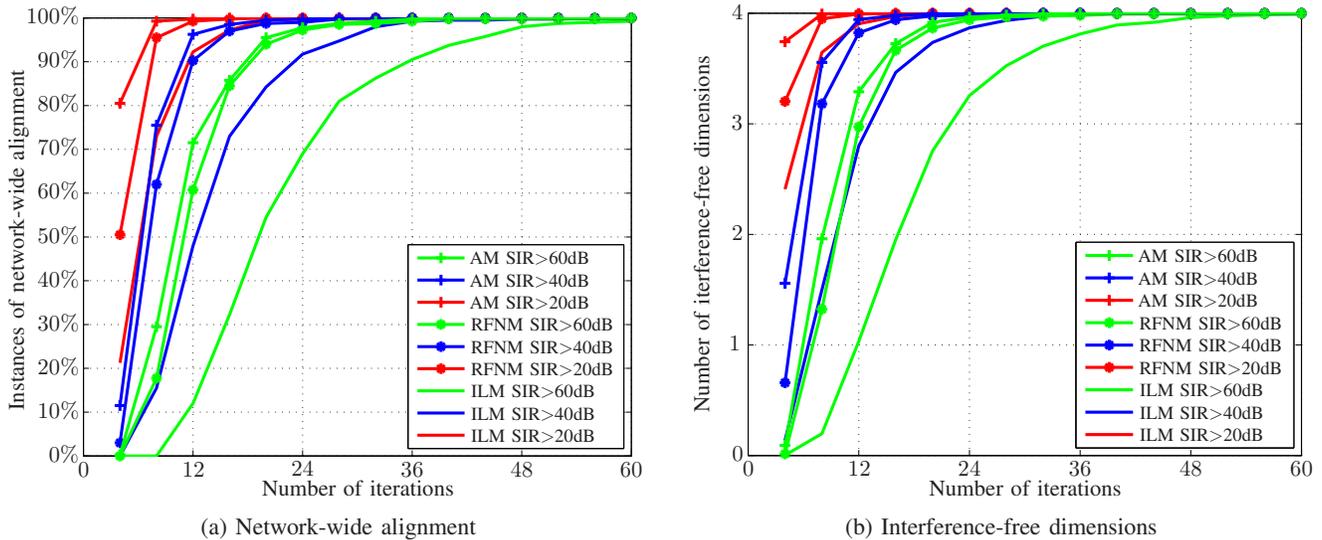


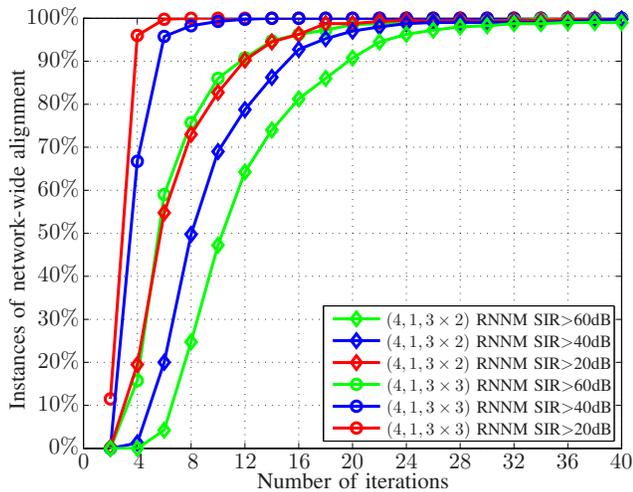
Fig. 2: Alignment in the  $(4, 1, 3 \times 3)$  network using the RFNM, AM and ILM algorithms.

as listed in Table I. All systems listed in Table I are chosen so as to lie on or below the proper-improper boundary ( $d \leq (M + N)/(GK + 1)$ ), and known to be feasible. We assume  $d = 1$  for each user in all cases, but different networks have different total number of data streams and need different number of interference-free dimensions at the BSs. Also note that for every system that is on the proper-improper boundary and has no redundant dimensions, we also consider an identical system with redundant antennas on either the user or the BS side.

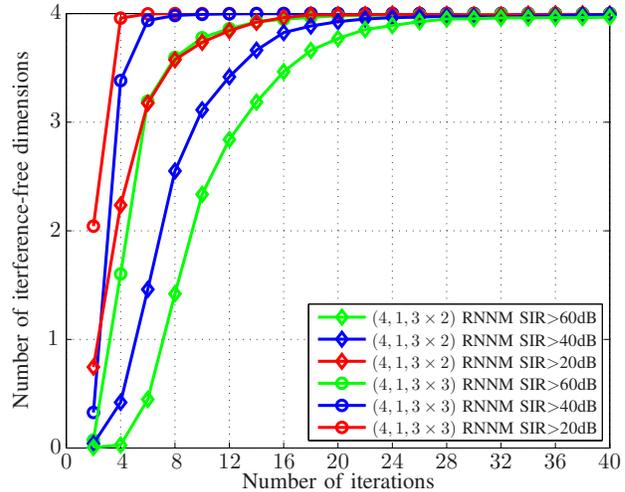
Since these algorithms are run on finite precision computers, a numerical threshold on interference suppression is required to declare interference alignment. We adopt signal-to-interference (SIR) ratio as a metric to evaluate interference suppression and declare interference to be aligned if SIR for every data stream in the network exceeds a certain threshold.

It is important to consider interference suppression across all data streams since there may exist several local minima that eliminate interference only in a subset of the streams. While a high SIR threshold is an accurate indicator of interference alignment, a lower threshold is of practical interest since most applications only require interference to be suppressed up to the noise floor. We consider SIR thresholds of 20dB, 40dB and 60dB. We opt to show the number of interference-free dimensions and instances of network-wide alignment rather than system-level performance metrics such as sum rate because the former is more directly related to the optimization objective of this paper.

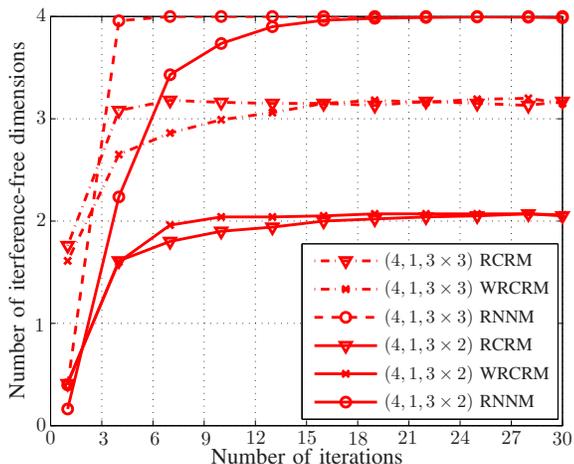
The algorithms are tested over at least 400 channel realizations with channel coefficients drawn from a complex circularly symmetric Gaussian distribution with unit variance in each dimension. We compare the algorithms developed



(a) Network-wide alignment



(b) Interference-free dimensions

 Fig. 3: Alignment in the  $(4, 1, 3 \times 2)$  and  $(4, 1, 3 \times 3)$  networks using the RNNM algorithm.

 Fig. 4: Interference-free dimensions in the  $(4, 1, 3 \times 2)$  and  $(4, 1, 3 \times 3)$  networks using the WRCRM, RCRM and RNNM algorithms. The SIR threshold is set to 20 dB.

in this paper with ILM proposed in [8], the RCRM formulation proposed in [11] and the reweighted nuclear norm minimization approach proposed to solve RCRM (WRCRM) in [16]. The convex optimization problems involved in these algorithms are solved using CVX, a package for specifying and solving convex programs [38], [39]. All algorithms are run for a fixed number of iterations.

One update of the transmit and receive beamformers is considered to be one iteration of an algorithm. In the case of ILM, each iteration requires computing  $G(K+1)$  eigenvalue decompositions, while RCRM and WRCRM require solving two SDPs per iteration. Each iteration of AM and RFNM requires solving one unconstrained quadratic program while RNNM requires solving one SDP. Due to the significant differences in the per-iteration complexities, the results corresponding to ILM, AM and RFNM algorithms are presented

separately from the results for RNNM, RCRM and WRCRM.

We first present results pertaining to the simplest two networks in Table I — the  $(4, 1, 3 \times 2)$  and the  $(4, 1, 3 \times 3)$  networks.

#### A. The $(4, 1, 3 \times 2)$ and $(4, 1, 3 \times 3)$ networks

The performance of the ILM, AM and RFNM algorithms on the  $(4, 1, 3 \times 2)$  network is plotted in Fig. 1. Note that this system has no redundant antennas and requires one interference-free dimension at each BS. Fig. 1(a) plots the number of instances (channel realizations) where network-wide alignment occurs as a function of the number of iterations. Network-wide alignment requires the SIR in each of the four data streams to exceed a certain threshold. Fig. 1(b) plots the average number of interference-free data streams created as a function of the number of iterations. In this case, we do not look for network-wide alignment and only focus on the SIR of each individual data stream. Scenarios where partial interference alignment occurs (interference eliminated in only some of the data streams) contribute towards interference-free dimensions (Fig. 1(b)) but not towards network-wide alignment (Fig. 1(a)). It is seen from Fig. 1(a) that both AM and RFNM show better performance than ILM. For example, after 120 iterations of each algorithm, RFNM achieves an SIR exceeding 60dB in all data streams for 65% of the channel realizations, but ILM only achieves an SIR exceeding 60dB in 40% of the channel realizations. It is observed that the performance difference between the different algorithms is minimal if the algorithms are each run for more than 600 iterations. Thus, the main advantage of AM and RFNM over ILM is in convergence speed rather than ultimate performance.

The  $(4, 1, 3 \times 3)$  network has a redundant antenna at each user and its impact is clearly seen in Fig. 2. ILM, AM and RFNM all converge to aligned solutions within 50-60 iterations, with AM and RFNM significantly outperforming ILM. The single additional antenna on the user-side reduces the number of iterations required by almost a factor of 10,

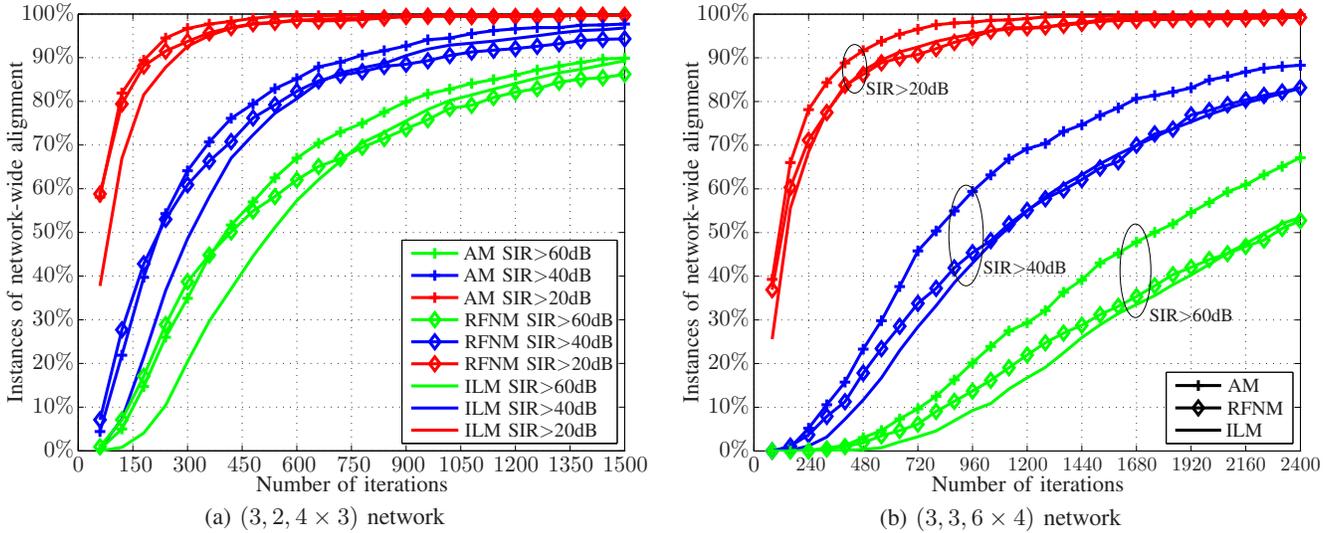


Fig. 5: Network-wide alignment using the RFNM, AM and ILM algorithms for two networks without redundant antennas.

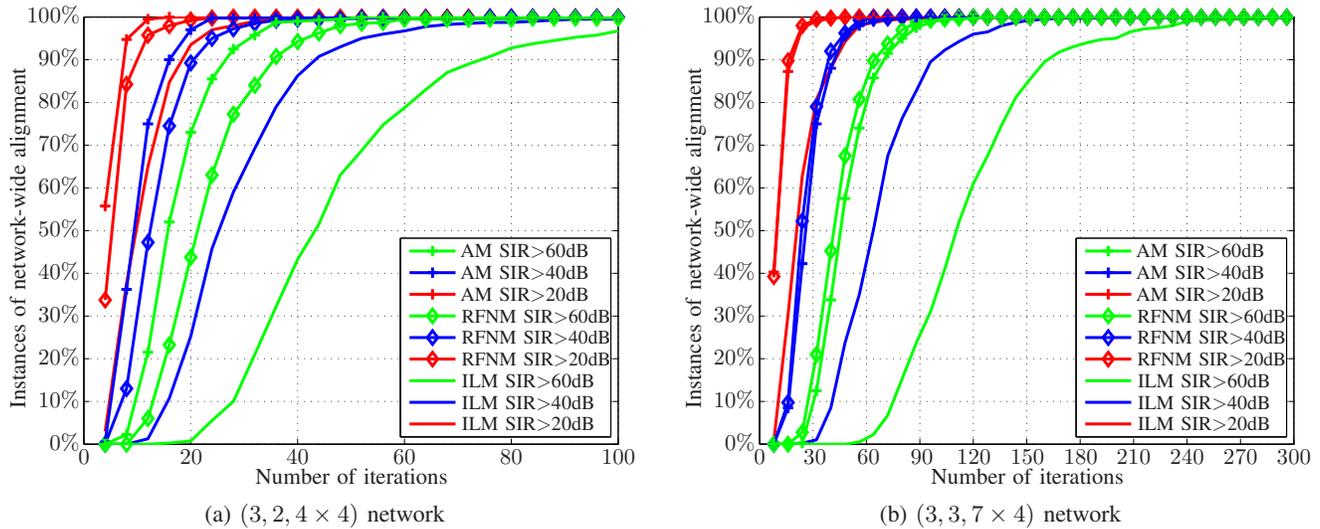


Fig. 6: Network-wide alignment using the RFNM, AM and ILM algorithms for two networks with redundant antennas.

thus highlighting the enormous value of redundant antennas when designing aligned beamformers.

Fig. 3 plots the performance of RNNM for the  $(4, 1, 3 \times 3)$  and  $(4, 1, 3 \times 2)$  networks. It is seen that aligned beamformers are designed within 10-20 iterations in all cases. The performance comparison of RCRM, WRCRM and RNNM is plotted in Fig. 4. In Fig. 4, we set the SIR threshold to 20dB and plot the number of interference-free dimensions generated. In these two networks, a maximum of four such dimensions can be created. It is seen that both RCRM and WRCRM do not generate the requisite number of interference-free dimensions and hence do not achieve network-wide interference alignment. It appears that within the first few iterations, these algorithms get trapped in local minima that do not reduce a sufficient number of singular values to zero. Although redundant antennas help in finding more interference-free dimensions, the RCRM and WRCRM algorithms still fall short of achieving network-wide interference alignment. This highlights the importance

of designing algorithms that are able to exploit the prior knowledge of the required rank.

### B. Larger Networks

To further test the observations made in the previous section we now consider larger networks where more interference-free dimensions are required at each BS. Fig. 5 plots the performance of ILM, AM and RFNM on the second and third set of networks listed in Table I that do not have redundant antennas. A broad trend that can be observed from Fig. 5 is that as the network size increases, all three algorithms need several thousand iterations to design aligned beamformers. Further, the performance gap between AM, RFNM and ILM grows smaller as the network size increases.

Fig. 6 considers the same two networks but now appended with one redundant antenna at either the BS or the user side. It is seen from Fig. 6 that the redundant antenna significantly

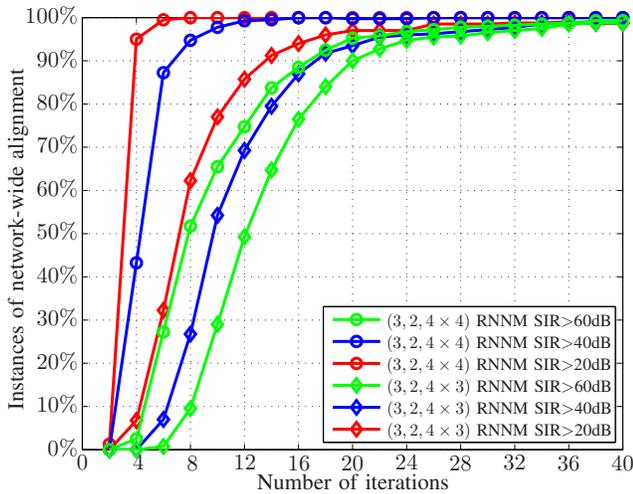
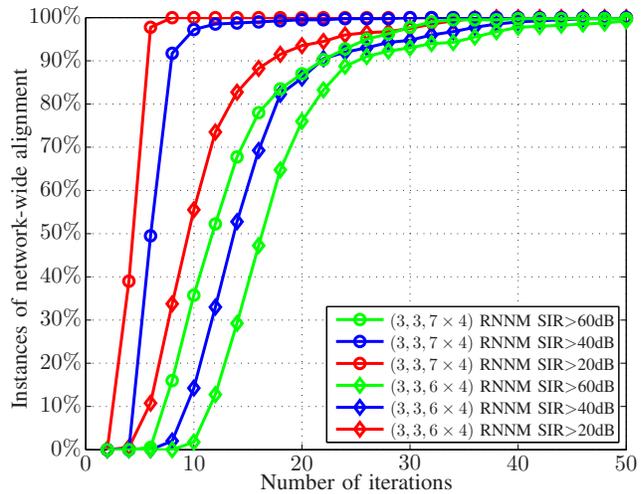
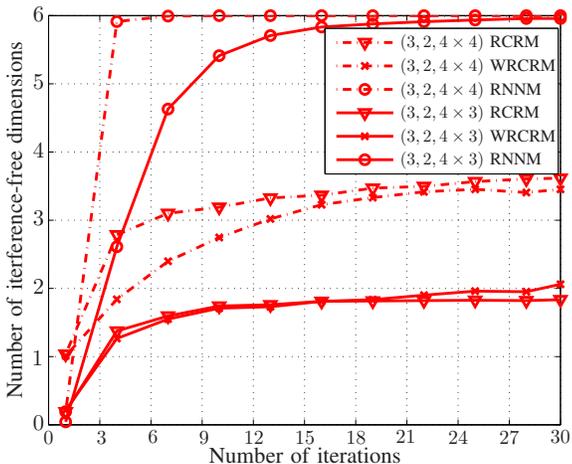

 (a)  $(3, 2, 4 \times 3)$  and  $(3, 2, 4 \times 4)$  network

 (b)  $(3, 3, 6 \times 4)$  and  $(3, 3, 7 \times 4)$  networks

Fig. 7: Network-wide alignment using RNNM.


 Fig. 8: Interference-free dimensions in the  $(3, 2, 4 \times 3)$  and  $(3, 2, 4 \times 4)$  networks using the WRCRM, RCRM and RNNM algorithms. The SIR threshold is set to 20 dB.

reduces the the number of iterations required to design aligned beamformers. In particular, AM and RFNM perform significantly better than ILM and require up to 25% fewer iterations for a similar level of performance.

Fig. 7 plots the performance of RNNM for the same two networks with and without redundant antennas. It is seen that unlike the previous three algorithms, RNNM is able to generate aligned beamformers, even for high SIR thresholds within about 20-40 iterations, irrespective of whether there are redundant antennas or not. There is only a marginal increase in the number of iterations required as the network size grows. As before, we see in Fig. 8 that the RCRM and WRCRM algorithms do not achieve the necessary number of interference-free dimensions, which is six this case.

A conclusion drawn from these simulations is that systems with redundant antennas permit the use of algorithms that

are not necessarily well suited for sparsity but have the advantage of having low per-iteration complexity. When this additional flexibility is absent, algorithms such as RNNM that are designed specifically to generate a desired level of sparsity are necessary to design aligned beamformers. Hence, this trade-off between redundant antennas and algorithmic complexity must be factored into design considerations for multi-antenna wireless networks.

## VIII. CONCLUSION

In this paper we propose several new approaches to designing aligned beamformers in a cellular network based on a reformulation of the conditions for interference alignment. Using these alternate conditions, we formulate a rank minimization problem to design aligned transmit beamformers in the uplink and solve the rank minimization using reweighted matrix norm minimization leading to RNNM and RFNM algorithms. A crucial aspect of these algorithms is a novel reweighting rule that exploits prior knowledge of the required rank of interference matrices.

This paper also develops the AM algorithm for interference alignment where the prior knowledge of the rank of the interference matrices is used to allow the rank-deficient matrices to be expressed as a product of two matrices, followed by optimizing these matrices in an alternating manner to minimize a quadratic objective. Each iteration of this algorithm only requires solving an unconstrained convex quadratic program. In terms of complexity, while RNNM requires solving an SDP in each iteration, RFNM and AM only require solving an unconstrained quadratic program.

Simulation results indicate that RNNM is very effective in designing aligned beamformers irrespective of the presence of redundant antennas. It significantly outperforms previously proposed RCRM and WRCRM algorithms (with similar per-iteration complexity) in terms of maximizing the number of interference-free dimensions. In contrast, RFNM and AM

show significantly faster convergence in systems with redundant antennas when compared to ILM (which has a similar per-iteration complexity).

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