Multiplexing Gain of Modulating Phases Through Reconfigurable Intelligent Surface

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***Abstract*—This paper investigates the information theoretical limit of a reconfigurable intelligent surface (RIS) aided communication scenario in which the RIS and the transmitter jointly send information to the receiver. The RIS is an emerging technology that uses a large number of passive reflective elements with adjustable phases to intelligently reflect the transmit signal to the intended receiver. While most previous studies of the RIS focus on its ability to beamform and to boost the received signal-to-noise ratio (SNR), this paper shows that if the information data stream is available both at the transmitter and the RIS and the phases at the RIS can be used to modulate data, then the *multiplexing gain* of the overall channel can potentially be significantly enhanced. Specifically, we show that in a multiple-input multiple-output (MIMO) channel with M transmit antennas and K receive antennas, a RIS with N reflective elements can improve the multiplexing gain from $\min(M, K)$ to $\min(M + \frac{N}{2} - \frac{1}{2}, N, K)$. This result is obtained by establishing a connection between the RIS system and the MIMO channel with phase noises and using results for characterizing the information dimension under projection.**

I. Introduction

Reconfigurable intelligent surface (RIS), also known as the intelligent reflective surface (IRS), is a promising new technology that utilizes a large number of reflective elements with adjustable phase shifts to enhance the spectral and energy efficiencies of wireless communication systems [1]–[4]. In most current literature, the RIS is envisioned to be used as a passive beamformer with its reflective coefficients optimized for maximizing the received signal-to-noise ratio (SNR) of the overall channel. Passive beamforming, however, does not fully harvest the potentials of the RIS. The goal of this paper is to show that if the information data stream is available at both the transmitter and the RIS, then the RIS can cooperate with the transmitter to potentially improve the multiplexing gain of the overall channel. In effect, the adjustable reflective coefficients can be used not only to enhance the channel, but also to carry information.

RIS can be thought of as an array of a large number of reflective elements, each of which can induce a phase shift between the incident signal and the reflecting signal. By adjusting the phase shifts, the RIS can enhance the transmission quality of the overall channel by adaptively beamforming the incident signal to an intended reflecting direction [5]. As compared to traditional relaying techniques [6], [7], the RIS reflects the incident signal passively, so it has much lower energy consumption. Further, by utilizing a large number of analog reflective elements and by adjusting their phases in real time, the RIS can achieve a high beamforming gain at relatively low hardware cost.

The existing literatures on the RIS-aided wireless communication system mostly focus on the optimization of the phase shifts at the RIS to achieve various system-level objectives [8]–[15]. These existing studies, however, utilize the RIS only as a passive analog beamformer that can adjust the phases of the reflection to achieve a beamforming gain. Passive analog beamforming works by adjusting the phase shifts of scattered wavefronts at different reflective elements so that they can add constructively at the receiver, thereby enhancing the receive SNR. For these systems, the phase shifts at the RIS are functions of the instantaneous channel realizations only.

The main focus of this paper is to explore the possibility that if, in addition to the channel state information (CSI), the RIS also has access to the user data stream, then the RIS can modulate the information stream in the phase shifts to significantly further enhance the capacity of the overall communication channel. This idea has already been explored in [16], where the achievable rate for the joint information transmission scheme is studied for a system with fixed finite modulation, and in [17] and [18], which address the design of decoding algorithms when extra information bits are encoded in the RIS, but the information theoretical limit of modulating through RIS is not yet known. Toward this end, this paper partially fills this gap by studying the degree of freedom (DoF), also known as pre-log factor or multiplexing gain, of a joint transmission scenario, in which the information bits are carried both in the transmit radio frequency (RF) signal and the phase shifts at the RIS for an intended MIMO receiver.

The multiplexing gain is an important metric which captures the high SNR asymptotic behavior of channel capacity for channels with additive Gaussian noises. The difficulty in the analysis for the RIS channel lies in the multiplicative nature of the channel model in which the received signal is the product of two different information-carrying signals. We tackle this challenge by connecting the achievable rate of the RIS channel to the capacity of the MIMO channel with phase noises. Tools for studying the information dimension and the point-wise dimension under projection are used. We show that for a system with M transmit antennas and K receive antennas, the use of an RIS with N reflective elements can improve the multiplexing gain of the overall channel from

$\min(M, K)$ to $\min(M + \frac{N}{2} - \frac{1}{2}, N, K)$. Thus, when there are more receive antennas than transmit antennas, a cooperative RIS can significantly improve the overall channel capacity by modulating additional data streams through its phase shifts.

II. System Model

Consider an RIS-aided communication scenario as shown in Fig. 1, where an RIS with N reflective elements is deployed to enhance the signal transmission from an M-antenna transmitter to a K-antenna receiver. Let $\mathbf{G}, \mathbf{H}$ be the channel response matrices from the transmitter to the RIS and from the RIS to the receiver, respectively, which are assumed to be fixed and known to both the transmitter and the receiver. We assume that each element of the RIS combines all the transmitted signals at a single point and reflects it from this point to the receiver with an adjustable phase shift. We assume that there is no direct path from the transmitter to the receiver. In this case, the discrete-time channel model for the communication scenario is given by

$$\mathbf{Y} = \sqrt{P}\mathbf{H}\underbrace{\mathbf{\Theta}\mathbf{G}\mathbf{X}}_{\mathbf{W}} + \mathbf{Z}, \tag{1}$$

where $\mathbf{Y} \in \mathbb{C}^K$ is the received signal vector, $\mathbf{X} \in \mathbb{C}^M$ is the transmit signal vector with power constraint $\mathbb{E}[\|\mathbf{X}\|_2^2] \leq 1$, and $\mathbf{Z} \sim \mathcal{CN}(0, \sigma^2 \mathbf{I}_K)$ is the additive Gaussian noise vector at the receiver. Here, $\mathbf{\Theta} = \mathrm{diag}([e^{j\theta_1}, e^{j\theta_2}, \cdots, e^{j\theta_N}])$ is the $N \times N$ reflecting coefficient matrix induced by the RIS, where θ_m is the phase-shifting coefficient of the m-th reflective element on the RIS and $\mathrm{diag}(\cdot)$ denotes a diagonal matrix whose diagonal elements are given by the corresponding entries of the argument. For convenience, we define $\mathbf{W} \triangleq \mathbf{\Theta}\mathbf{G}\mathbf{X}$ as shown in (1) so that the output of the RIS is $\sqrt{P}\mathbf{W}$.

When the RIS is used as a passive beamformer, we design a fixed $\mathbf{\Theta}$ as a function of $\mathbf{H}$ and $\mathbf{G}$ in order to maximize the over channel capacity from $\mathbf{X}$ to $\mathbf{Y}$. Mathematically, this maximization operation can be written as

$$\max_{\mathbf{\Theta}} \max_{p(\mathbf{x})} I\left(\mathbf{X}; \mathbf{Y} | \mathbf{\Theta}\right). \tag{2}$$

In this case, the reflecting coefficients do not carry any information. The main function of the RIS is to enhance the beamforming gain from $\mathbf{X}$ to $\mathbf{Y}$.

In this paper, we explore the potential of using the RIS to encode information in the reflective coefficients themselves. In this case, the channel is defined to be from $(\mathbf{X}, \mathbf{\Theta})$ to $\mathbf{Y}$. The capacity of such a channel, as a function of the transmitted power P, can be written as

$$C(P) = \max_{p(\mathbf{x}, \theta)} I(\mathbf{X}, \mathbf{\Theta}; \mathbf{Y}). \tag{3}$$

Computing $I(\mathbf{X}, \mathbf{\Theta}; \mathbf{Y})$ is not an easy task as it involves a multi-dimensional integration over the probability density functions (pdf) of $\mathbf{X}$ and $\mathbf{\Theta}$ and can only be done numerically. To obtain some insights into the problem, in this paper, we characterize the multiplexing gain m of this channel model, which is defined as the pre-log factor of the rate as $P \to \infty$, i.e.

$$m \triangleq \lim_{P \to \infty} \frac{C(P)}{\log P}. \tag{4}$$

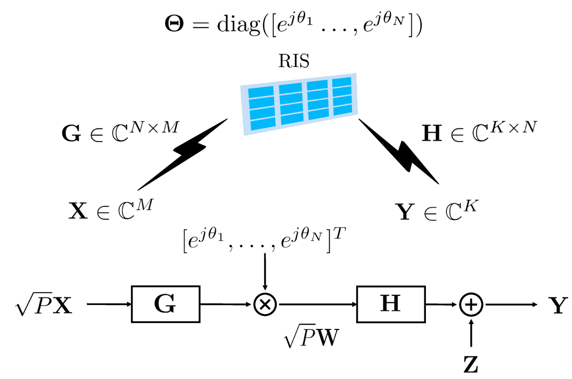

Fig. 1. Channel model for an RIS assisted point-to-point MIMO communication system.

The multiplexing gain, also known as degree-of-freedom, characterizes the number of independent data streams that can be transmitted from the transmitter to the receiver. Understanding this high SNR behavior guides the design of practical communication systems.

For a point-to-point MIMO channel with M transmit antennas and K receive antennas, the maximum multiplexing gain is simply $\min(M, K)$. The main goal of this paper is to characterize the improvement in the multiplexing gain when an N-element RIS is deployed between the transmitter and the receiver.

III. Multiplexing Gain

In this section, we compute the multiplexing gain of $I(\mathbf{X}, \mathbf{\Theta}; \mathbf{Y})$ based on an equivalence between the multiplexing gain of an additive noise channel and the information dimension of the input. Toward this end, we first compute the information dimension of the output $\mathbf{W}$ at the RIS. This is accompolished by building a connection between the channel model with RIS and the MIMO channel with phase noise. We then compute the information dimension of $\mathbf{H}\mathbf{W}$ by studying the behavior of information dimension under projection.

A. Information Dimension and Multiplexing Gain

First, we introduce the concept of information dimension and the connection between the information dimension and the multiplexing gain of a channel with additive noise.

Definition 1: For a random vector $\mathbf{X} \in \mathbb{R}^N$ with distribution μ, we define the information dimension of $\mathbf{X}$ as

$$D(\mathbf{X}) = \lim_{\varepsilon \to 0} \frac{\mathbb{E}\left[\log \mu\left(B(\mathbf{X}; \varepsilon)\right)\right]}{\log \varepsilon} \tag{5}$$

assuming that the limit exists, where $B(\mathbf{X}; \varepsilon) \subseteq \mathbb{R}^N$ denotes the ball with center $\mathbf{X}$ and radius ε with respect to an arbitrary norm on $\mathbb{R}^N$.

The following lemma connects the information dimension and the multiplexing gain of a general vector additive noise channel.

Lemma 1 ([19], [20]): Let $\mathbf{X}$ and $\mathbf{Z}$ be independent random vectors in $\mathbb{R}^n$ such that $\mathbf{Z}$ has an absolutely continuous distribution with $h(\mathbf{Z}) > -\infty$ and $H(\lfloor \mathbf{Z} \rfloor) < \infty$. Then

$$\limsup_{P\to\infty} \frac{I(\mathbf{X}; \sqrt{P}\mathbf{X}+\mathbf{Z})}{\frac{1}{2}\log P} = D(\mathbf{X}). \tag{6}$$

By Lemma 1, we see that computing the multiplexing gain of the RIS channel (1) can be equivalently cast as computing the information dimension $D(\mathbf{H\Theta GX})$. However, Lemma 1 deals with real channels. To work with a complex channel model, we can stack the real and imaginary parts of the complex vectors, and stack the real and imaginary parts of complex matrices, such as $\mathbf{H}$, as

$$\begin{bmatrix} \mathrm{Re}\{\mathbf{H}\} & \mathrm{Im}\{\mathbf{H}\} \\ \mathrm{Im}\{\mathbf{H}\} & \mathrm{Re}\{\mathbf{H}\} \end{bmatrix}, \tag{7}$$

then apply Lemma 1 to the resulting equivalent model in $\mathbb{R}^{2K}$. Note that two real dimensions in $\mathbb{R}^{2K}$ is equivalent to one complex dimension in $\mathbb{C}^K$.

B. Information Dimension of $\mathbf{W}$

Next, we compute the information dimension of the output of the RIS $\mathbf{W}$ by looking at the following channel model (with general arbitrary dimensions):

$$\tilde{\mathbf{Y}} = \sqrt{P}\tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}} + \tilde{\mathbf{Z}}, \tag{8}$$

where $\tilde{\mathbf{Y}} \in \mathbb{C}^{\tilde{N}}$, $\tilde{\mathbf{X}} \in \mathbb{C}^{\tilde{M}}$, $\tilde{\mathbf{\Theta}} = \mathrm{diag}([e^{j\theta_1}, e^{j\theta_2}, \cdots, e^{j\theta_{\tilde{N}}}])$. A crucial observation is that this channel model resembles the MIMO channel with phase noise. We leverage the following result from the phase noise literature to provide a key insight.

Lemma 2 ([21]): Let $\tilde{\mathbf{\Phi}} = [e^{j\theta_1}, \ldots, e^{j\theta_{\tilde{N}}}]^T$ be such that $h(\tilde{\mathbf{\Phi}}) > -\infty$ and

$$\bar{\mathbf{Y}} = \sqrt{P}\tilde{\mathbf{\Phi}} \circ \left(\tilde{\mathbf{G}}\tilde{\mathbf{X}}\right) + \tilde{\mathbf{Z}}, \tag{9}$$

where $\tilde{\mathbf{Z}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{N}})$. For $\tilde{\mathbf{X}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}})$, we have that

$$h(\bar{\mathbf{Y}}) \geq \min\left(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}\right) \log^+ P + c_1. \tag{10}$$

Further, for any distribution of $\mathbf{X}$, we have

$$I(\bar{\mathbf{Y}}; \tilde{\mathbf{X}}) \leq \min\left(\tilde{M} - \frac{1}{2}, \frac{\tilde{N}}{2}\right) \log^+ P + c_2 \tag{11}$$

Here c_1 and c_2 are constants that do not depend on P.

The intuition behind (10) is the following. The information dimension of the entropy term $h(\bar{\mathbf{Y}})$ is clearly lower bounded by the dimension of $\bar{\mathbf{Y}}$ which is $\tilde{N}$, but it is also lower bounded by the transmit dimension, which is the sum of $\tilde{M}$ and $\frac{\tilde{N}}{2}$ but subtracting $\frac{1}{2}$. Here, $\tilde{M}$ is the information dimension of $\tilde{\mathbf{X}}$; $\frac{\tilde{N}}{2}$ is the information dimension of the phases $\tilde{\mathbf{\Phi}}$; but because there is a common phase between them as the two are multiplied, we need to subtract the dimension of their overlap, which is $\frac{1}{2}$.

We are now ready to characterize the information dimension of $\mathbf{W}$ in (1). The result below is for arbitrary dimensions $\tilde{N}$ and $\tilde{M}$.

Theorem 1: For $\tilde{\mathbf{Y}} \in \mathbb{C}^{\tilde{N}}$, $\tilde{\mathbf{X}} \in \mathbb{C}^{\tilde{M}}$, $\tilde{\mathbf{\Theta}} = \mathrm{diag}([e^{j\theta_1}, e^{j\theta_2}, \cdots, e^{j\theta_{\tilde{N}}}])$ and the channel model

$$\tilde{\mathbf{Y}} = \sqrt{P}\tilde{\mathbf{W}} + \tilde{\mathbf{Z}} = \sqrt{P}\tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}} + \tilde{\mathbf{Z}}, \tag{12}$$

where $\tilde{\mathbf{Z}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{N}})$. For almost all matrices $\tilde{\mathbf{G}} \in \mathbb{C}^{\tilde{N}\times\tilde{M}}$, the multiplexing gain of $I(\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}})$ is given by $\min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$. This multiplexing gain can be achieved with $\tilde{\mathbf{X}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}})$ and $\tilde{\mathbf{\Theta}}$ such that $\theta_i \sim U[-\pi, \pi]$. With this choice of distributions, the information dimension of $\tilde{\mathbf{W}}$ is

$$D(\tilde{\mathbf{W}}) = \min\left(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}\right). \tag{13}$$

Proof: We show the achievability by making use of Lemma 2. Observe that with $\tilde{\mathbf{\Phi}} = [e^{j\theta_1}, \ldots, e^{j\theta_{\tilde{N}}}]^T$, we have

$$\tilde{\mathbf{Y}} = \sqrt{P}\tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}} + \tilde{\mathbf{Z}} \tag{14}$$
$$= \sqrt{P}\tilde{\mathbf{\Phi}} \circ (\tilde{\mathbf{G}}\tilde{\mathbf{X}}) + \tilde{\mathbf{Z}}. \tag{15}$$

Therefore, the RIS channel model in (12) can be viewed as a channel with phase shifts $\tilde{\mathbf{\Phi}}$ applied at the receiver. Choosing $\tilde{\mathbf{X}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}})$ and $\tilde{\mathbf{\Phi}}$ as $\theta_i \sim U[-\pi, \pi]$, we have that $h(\tilde{\mathbf{\Phi}}) > -\infty$, therefore we can apply (10) in Lemma 2 to get $h(\tilde{\mathbf{Y}}) \geq \min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}) \log^+ P + c_1$.

Further, the mutual information can be lower bounded as:

$$I(\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}) = h(\tilde{\mathbf{Y}}) - h(\tilde{\mathbf{Y}}|\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}}) \tag{16}$$
$$\geq \min\left(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}\right) \log^+ P + c_1, \tag{17}$$

where the inequality comes from the fact that $h(\tilde{\mathbf{Y}}|\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}})$ is the entropy of the noise term which does not depend on P.

Next, we prove the converse by giving an upper bound on $I(\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}})$. First, for $\tilde{\mathbf{\Theta}}$ such that $h(\tilde{\mathbf{\Theta}}) > -\infty$

$$I\left(\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}\right) = I(\tilde{\mathbf{X}}; \tilde{\mathbf{Y}}) + I(\tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}|\tilde{\mathbf{X}}) \tag{18}$$
$$\leq \min\left(\tilde{M} - \frac{1}{2}, \frac{\tilde{N}}{2}\right) \log^+ P + I\left(\tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}|\tilde{\mathbf{X}}\right) + c_2, \tag{19}$$

where the inequality follows from applying (11) in Lemma 2. To upper bound $I\left(\tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}|\tilde{\mathbf{X}}\right)$, note that when $\tilde{\mathbf{X}}$ is known,

$$\tilde{\mathbf{Y}} = \sqrt{P}\tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}} + \tilde{\mathbf{Z}} \tag{20}$$
$$= \sqrt{P}\mathrm{diag}(\tilde{\mathbf{G}}\tilde{\mathbf{X}})\tilde{\mathbf{\Phi}} + \mathbf{Z}, \tag{21}$$

where $\tilde{\mathbf{\Phi}} = [e^{j\theta_1}, \ldots, e^{j\theta_{\tilde{N}}}]^T$. Thus, (20) is equivalent to a MIMO channel with continuous phase inputs on each of the transmit antennas, so

$$I\left(\tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}|\tilde{\mathbf{X}}\right) = I(\tilde{\mathbf{\Phi}}; \mathrm{diag}(\tilde{\mathbf{G}}\tilde{\mathbf{X}})\tilde{\mathbf{\Phi}} + \tilde{\mathbf{Z}}|\tilde{\mathbf{X}}) \tag{22}$$
$$= \sum_{i=1}^{\tilde{N}} I\left(\theta_i; \sqrt{P}(\tilde{\mathbf{G}}\tilde{\mathbf{X}})_i e^{j\theta_i} + \tilde{Z}_i|\tilde{\mathbf{X}}\right) \tag{23}$$
$$\leq \frac{\tilde{N}}{2}\log^+ P + c. \tag{24}$$

where the last equality follows from applying capacity results for phase-only transmission in single-antenna channels [22] and c is a constant that does not depend P. Now we have

$$I\left(\tilde{\mathbf{X}}, \tilde{\mathbf{\Theta}}; \tilde{\mathbf{Y}}\right) \leq \min\left(\tilde{M} - \frac{1}{2}, \frac{\tilde{N}}{2}\right) \log^+ P + \frac{\tilde{N}}{2} \log^+ P + c \tag{25}$$

$$= \min\left(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}\right) \log^+ P + c. \tag{26}$$

It is not hard to see that the above still holds even when $h(\tilde{\mathbf{\Theta}}) = -\infty$. This is because even if some θ_i is discrete, the $\min(\tilde{M} - \frac{1}{2}, \frac{\tilde{N}}{2})$ term in (25) due to the information dimension of $\tilde{\mathbf{X}}$ would increase by at most $\frac{1}{2}$, while the $\frac{\tilde{N}}{2}$ term due to the information dimension of $\tilde{\mathbf{\Theta}}$ would decrease by at least $\frac{1}{2}$.

Combining (17) and (26), we see that the multiplexing gain of the channel model (12) is given by $\min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$. The information dimension of $\tilde{\mathbf{W}} = \tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}}$ is therefore given by $D(\tilde{\mathbf{W}}) = \min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$ in view of Lemma 1. ■

The above result is established with the choice of $\tilde{\mathbf{X}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}})$ and $\tilde{\mathbf{\Theta}}$ such that $\theta_i \sim U[-\pi, \pi]$. But the final multiplexing gain expression in Theorem 1 indicates that there is $\frac{1}{2}$ dimension overlap between $\tilde{\mathbf{\Theta}}$ and $\tilde{\mathbf{X}}$ giving rise to the $-\frac{1}{2}$ term. This suggests that removing one of the phase angle from either $\tilde{\mathbf{X}}$ or $\tilde{\mathbf{\Theta}}$ would give the same multiplexing gain. This is formally stated in the following Proposition.

Proposition 1: For the channel model in Theorem 1, the same multiplexing gain of $\min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$ and thus the same information dimension $D(\tilde{\mathbf{W}}) = \min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$ can be achieved using the following choice of distributions for $\tilde{\mathbf{X}}'$ and $\tilde{\mathbf{\Theta}}'$. The distribution of $\tilde{\mathbf{X}}'$ is chosen such that the first $\tilde{M} - 1$ elements are i.i.d. complex Gaussian, i.e., $\tilde{\mathbf{X}}'_{\tilde{M}-1} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}-1})$, and $\tilde{X}'_{\tilde{M}}$ is chosen to be real with a chi-squared distribution with 2 degrees of freedom, and $\tilde{\mathbf{\Theta}}'$ remains as $\tilde{\theta}'_i \sim U[-\pi, \pi]$.

Proof: We start from the $\tilde{\mathbf{X}} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}})$ and $\tilde{\mathbf{\Theta}}$ with $\tilde{\theta}_i \sim U[-\pi, \pi]$ as in Theorem 1 for achieving the multiplexing gain for $\tilde{\mathbf{W}} = \tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}}$ of $\min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$. Denote the phase of the last element of $\tilde{\mathbf{X}}$ as $e^{j\phi}$. Now, we chose $\tilde{\mathbf{X}}'$ as $e^{-j\phi}\tilde{\mathbf{X}}$ and $\tilde{\mathbf{\Theta}}' = e^{j\phi}\tilde{\mathbf{\Theta}}$. We have that the distribution of $\tilde{\mathbf{\Theta}}'\tilde{\mathbf{G}}\tilde{\mathbf{X}}'$ is the same as the distribution of $\tilde{\mathbf{\Theta}}\tilde{\mathbf{G}}\tilde{\mathbf{X}}$, so the same multiplexing gain can be achieved for $\tilde{\mathbf{W}}' = \tilde{\mathbf{\Theta}}'\tilde{\mathbf{G}}\tilde{\mathbf{X}}'$ and also the same information dimension. For this choice of distributions, $\tilde{\mathbf{X}}'$ is such that the first $\tilde{M} - 1$ elements are i.i.d. complex Gaussian, i.e. $\tilde{\mathbf{X}}'_{\tilde{M}-1} \sim \mathcal{CN}(0, \mathbf{I}_{\tilde{M}-1})$, the real part of $X'_{\tilde{M}}$ is a chi-squared distribution with 2 degrees of freedom, and the imaginary part of $X'_{\tilde{M}}$ is 0, and $\tilde{\mathbf{\Theta}}'$ as $\tilde{\theta}'_i \sim U[-\pi, \pi]$. ■

C. Information Dimension of HW

We now characterize the multiplexing gain of the original channel model (1) by investigating the information dimension of $\mathbf{HW}$. Observe that $D(\mathbf{HW})$ is the information dimension of $\mathbf{W}$ under a linear projection from $\mathbb{C}^N \to \mathbb{C}^K$, so one would expect that

$$D(\mathbf{HW}) = \min(D(\mathbf{W}), K). \tag{27}$$

The above relationship is, however, not true for any arbitrary distribution; see [19] for a counterexample. The issue is that the projection may have different effects for different points if the distribution is not absolutely continuous. For the particular $\mathbf{W}$ studied in the context of this paper, the relationship (27) turns out to be true. To establish this rigorously, we utilize results about the information dimension under projection, but first we need to define the concept of point-wise dimension.

Definition 2: For every point $\mathbf{x}$ and a probability distribution μ, we define the point-wise dimension at $\mathbf{x}$ as

$$d(\mathbf{x}) = \lim_{\varepsilon \to 0} \frac{\log \mu\left(B(\mathbf{x}; \varepsilon)\right)}{\log \varepsilon} \tag{28}$$

assuming that the limit exists.

From the literature of fractal geometry, we have the following result ensuring that the point-wise dimension does not increase under projection.

Lemma 3 ([23]): Let μ be a probability measure in $\mathbb{R}^E$, for any smooth function $f : \mathbb{R}^E \to \mathbb{R}^D$. If $d(\mathbf{x})$ exists for almost every $\mathbf{x}$, then $d(f(\mathbf{x}))$ exists and

$$d(f(\mathbf{x})) \leq \min(d(\mathbf{x}), D), \tag{29}$$

for almost every $\mathbf{x}$.

A similar result also holds for the information dimension. Moreover, when the point-wise dimension is upper bounded by the dimension of the range space of the projection, the information dimension is perserved under projection.

Lemma 4 ([23]): Given a matrix $\mathbf{A} \in \mathbb{R}^{D \times E}$ and $\mathbf{X} \in \mathbb{R}^E$ following a distribution μ, we have

$$D(\mathbf{AX}) \leq \min(D(\mathbf{X}), D). \tag{30}$$

Additionally, if the point-wise dimension of μ exists for almost every $\mathbf{x}$ and $d(\mathbf{x}) \leq D$, then for almost all $\mathbf{A}$ the information dimension of $\mathbf{AX}$ exists and

$$D(\mathbf{AX}) = D(\mathbf{X}). \tag{31}$$

Now we are ready to state our main theorem for the information dimension of $\mathbf{HW}$:

Theorem 2: For almost all matrices $\mathbf{G} \in \mathbb{C}^{N \times M}$ and $\mathbf{H} \in \mathbb{C}^{K \times N}$, the multiplexing gain of the channel model (1), i.e.,

$$\mathbf{Y} = \sqrt{P}\mathbf{HW} + \mathbf{Z} = \sqrt{P}\mathbf{H\Theta GX} + \mathbf{Z}, \tag{32}$$

with $(\mathbf{X}, \mathbf{\Theta})$ as the input, and equivalently the information dimension of $\mathbf{HW}$ at the optimal input distribution, are

$$m = D(\mathbf{HW}) = \min\left(M + \frac{N}{2} - \frac{1}{2}, N, K\right). \tag{33}$$

Proof: First, we establish that (33) is an upper bound for $D(\mathbf{HW})$. From Theorem 1 and Lemma 1 we have that $D(\mathbf{W}) \leq \min(M + \frac{N}{2} - \frac{1}{2}, N)$ for any distribution of $\mathbf{\Theta}$ and $\mathbf{X}$. Therefore, by (30) in Lemma 4, we have $D(\mathbf{HW}) \leq \min(M + \frac{N}{2} - \frac{1}{2}, N, K)$.

To establish the lower bound, i.e., $D(\mathbf{HW}) \geq \min(M + \frac{N}{2} - \frac{1}{2}, N, K)$, we divide into two cases and first treat the case when $K \geq \min(M + \frac{N}{2} - \frac{1}{2}, N, K)$. The key idea is to recognize that because $\mathbf{X}$ and $\mathbf{\Theta}$ have $\frac{1}{2}$ dimension overlap, we should choose the distributions for $(\mathbf{X}, \mathbf{\Theta})$ as that of $(\tilde{\mathbf{X}}', \tilde{\mathbf{\Theta}}')$ in Proposition 1 with $\tilde{M} = M$ and $\tilde{N} = N$ in order to explicitly account for the overlap. In this case, $(\mathbf{X}, \mathbf{\Theta})$ has an absolutely continuous distribution of dimension $M + \frac{N}{2} - \frac{1}{2}$, so it has a point-wise dimension $d(\mathbf{X}, \mathbf{\Theta}) = M + \frac{N}{2} - \frac{1}{2}$ for every point. Now, denote the mapping from $(\mathbf{X}, \mathbf{\Theta})$ to $\mathbf{W}$ as f, which is smooth. Then from Lemma 3, we have the following upper bound on the point-wise dimension

$$\begin{aligned} d(\mathbf{W}) &= d(f(\mathbf{X}, \mathbf{\Theta})) && (34) \\ &\leq \min(d(\mathbf{X}, \mathbf{\Theta}), N) = \min\left(M + \frac{N}{2} - \frac{1}{2}, N\right). && (35) \end{aligned}$$

For the case of $K \geq \min(M + \frac{N}{2} - \frac{1}{2}, N)$, we can now apply the second part of Lemma 4 to obtain

$$D(\mathbf{HW}) = D(\mathbf{W}) = \min\left(M + \frac{N}{2} - \frac{1}{2}, N\right). \tag{36}$$

For the case of $K < \min(M + \frac{N}{2} - \frac{1}{2}, N)$, the idea is to choose $\tilde{M} \leq M$ and $\tilde{N} \leq N$ such that $K = \min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N})$ and use such $(\tilde{M}, \tilde{N})$ in Theorem 1. In effect, we are choosing a distribution for $(\mathbf{X}, \mathbf{\Theta})$ by turning off some of the transmit antennas and reflecting elements. Next, we apply the same argument as in the case of $K \geq \min(M + \frac{N}{2} - \frac{1}{2}, N)$, which gives

$$\begin{aligned} d(\mathbf{W}) &= d(f(\mathbf{X}, \mathbf{\Theta})) && (37) \\ &\leq \min(d(\mathbf{X}, \mathbf{\Theta}), \tilde{N}) = \min\left(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}\right). && (38) \end{aligned}$$

Then, the proof follows by applying Lemma 4 to obtain that $D(\mathbf{HW}) = D(\mathbf{W}) = \min(\tilde{M} + \frac{\tilde{N}}{2} - \frac{1}{2}, \tilde{N}) = K$ is achievable.

Combining the two cases gives us the achievability. Together with the converse and by making use of Lemma 1, we obtain that the multiplexing gain of (1) is given by

$$m = D(\mathbf{HW}) = \min\left(M + \frac{N}{2} - \frac{1}{2}, N, K\right). \tag{39}$$

■

IV. Interpretations and Practical Implications

We now give an interpretation of the main multiplexing gain result (33). The channel model has two inputs: $\mathbf{X}$ of dimension M, and $\mathbf{\Theta}$ of dimension N. But because $\mathbf{\Theta}$ has phase control only, its information dimension is only $\frac{N}{2}$. Further, because $\mathbf{X}$ and $\mathbf{\Theta}$ have $\frac{1}{2}$ dimension overlap, the total input dimension is therefore $M + \frac{N}{2} - \frac{1}{2}$. Moreover, the channel has output $\mathbf{Y}$ of dimension K, and the RIS acts as a relay with a bottleneck $\mathbf{W}$ of dimension N. The overall multiplexing gain must be the minimum of the input dimension, the relay dimension, and the output dimension. This is why the multiplexing gain is $\min(M + \frac{N}{2} - \frac{1}{2}, N, K)$.

As compared to a MIMO channel with M antennas at the input and K antennas at the output achieving a multiplexing gain of $\min(M, K)$, an RIS system can improve the multiplexing gain for the case of $K > M$. In essence, the phase shifts at the RIS can act as part of the input to effectively increase the total input dimensions. Thus, by making the input data available to the RIS, we are enabling the RIS to modulate additional data stream via phase shifts. Assuming that N is much larger than M and K as in typical deployment scenarios, the maximum additional number of data streams that the RIS can provide is $\frac{N}{2} - \frac{1}{2}$. Note that if the RIS is used merely as a beamformer or as a reflector, it cannot improve the multiplexing gain. Thus, modulating data through phase shifts has significant advantage.

The result of this paper continues to hold even in the extreme case where the transmitter sends only a constant signal (which carries no information, thus $M = 0$) and all the information is carried in the phases of the RIS [24], [25]. In this case $I(\mathbf{X}, \mathbf{\Theta}; \mathbf{Y}) = I(\mathbf{\Theta}; \mathbf{Y}|\mathbf{X})$. The multiplexing gain that can be achieved is $\min(\frac{N}{2}, K)$. Therefore, to achieve full multiplexing gain of K, we need at least $N = 2K$.

Another practical case is when the transmitter has a single radio-frequency (RF) chain. This is akin to using the RIS to emulate a MIMO transmitter, i.e., a single-antenna active transmitter together with an RIS can be jointly configured to act as a MIMO array [26]. Such a system can be considered as a low-cost alternative to a fully digital MIMO system. In this case, we have $M = 1$ and the signal model is given by

$$\mathbf{Y} = \mathbf{H\Theta g}X + \mathbf{W}, \tag{40}$$

Assuming a large N, the main result of this paper implies that the overall multiplexing gain is K. One possible way for achieving this multiplexing gain is to use the amplitude of X to modulate $\frac{1}{2}$ DoF and to use a RIS of size $N = 2K - 1$ to modulate the remaining $(K - \frac{1}{2})$ DoF.

V. Conclusion

This paper studies the information theoretic limits of joint information transmission using both the RF input and phase modulation at an RIS reflector. In particular, the multiplexing gain of the communication schemes where information is conveyed through both the RF transmitted symbols and the reflective coefficients of the RIS is characterized. This is in contrast to most conventional use of the RIS as a passive beamformer for boosting the channel strength. The multiplexing gain is computed by recognizing a connection to the MIMO channels with phase noises. Tools for studying the information dimension and the point-wise dimension under projection are used to obtain the final result. We show that for a system with M transmit antennas and K receive antennas, the use of an RIS with N reflective elements can improve the multiplexing gain of the overall channel from $\min(M, K)$ to $\min(M + \frac{N}{2} - \frac{1}{2}, N, K)$.

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