# Gaussian Broadcast Channels with Bidirectional Conferencing Decoders and Correlated Noises

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***Abstract*—The two-user Gaussian broadcast channel (BC) with correlated noises and with decoders connected by cooperative links of finite capacities (known as conferencing decoders) is considered. A novel outer bound on the capacity region is established. For the channel with fully correlated noises (i.e., the noise correlation is either 1 or -1), the new outer bound yields exact capacity region for two cases: 1) BCs with degraded message sets; 2) BCs with one-sided conferencing from the weaker receiver to the stronger receiver. For these two cases, it is also shown that the outer bound is within half bits to the capacity region for arbitrary noise correlation. Furthermore, for the Gaussian BC with arbitrary noise correlation $\lambda$, we show that regardless of the capacities of conferencing links, a one-sided cooperative scheme (from the stronger user to the weaker one) based on decode-and-forward is sufficient to achieve the capacity region to within $\frac{1}{2}\log(\frac{2}{1-|\lambda|})$ bits.**

## I. Introduction

This paper investigates the impact of decoder cooperation via digital links of finite capacities (known as conferencing links) on the capacity region of a two-user Gaussian broadcast channel (BC) with both common and private messages and with correlated noises. The conferencing links can help the decoding process by providing a quantized version of the received signal or a part of the decoded messages from one receiver to the other receiver. However, among the existing results in the literature, it is rarely the case that a combination of the quantize-and-forward and decode-and-forward strategies can achieve the capacity region of any instances of BC with receiver cooperation. In this paper, we take advantage of a novel outer bound to show that in the case of fully correlated noises, exact capacity region results can be derived in several cases of Gaussian BC with fully correlated noises at the receivers and conferencing decoders. Interestingly, the conferencing links between the receivers with fully correlated noises can allow strictly positive rates to be achieved with near-zero transmit power when the noises are fully correlated.

The two-user BC with conferencing decoders is previously studied in [1]- [7]. In [1], the authors develop communication strategies for the interactive decoding of a common message broadcast to cooperative users. In [2], the capacity region of physically degraded channel is derived and also an achievable rate region is given for the general case. In [3], problems of communication over physically degraded, state-dependent BCs with one-sided conferencing decoders are investigated. In [4], the capacity region of the semi-deterministic BC with one-sided decoder cooperation is derived and its duality with a source coding problem is addressed. The authors in [5] consider the BC with one-sided cooperating users under strong secrecy constraints and present capacity results for the semi-deterministic and physically degraded cases. In [6], the BC with (one-sided) unreliable cooperating decoders is studied. In a recent work [7], the BC with degraded message sets and one-sided cooperation link that may be absent is considered and its capacity region is given. As reviewed, most of the previous capacity results on the BC with conferencing decoders pertain to one-sided cooperation, i.e., only one of the users is connected to the other by a conferencing link.

Following the previous work [8], in this paper, we first present a novel outer bound on the capacity region of the two-user Gaussian BC with bidirectional conferencing decoders. This outer bound, which is based on multiple applications of the Csiszár-Körner identity [9, Lemma 7] and the entropy power inequality, is strictly tighter than the previous ones including that of [2, Prop.1], which is essentially the cut-set bound. Using this bound, we prove several interesting results. For the channel with fully correlated noises (i.e., the noise correlation is either 1 or -1), the new outer bound yields exact capacity region for two cases: 1) Gaussian BCs with degraded message sets; 2) Gaussian BCs with one-sided conferencing from the weaker receiver to the stronger receiver. For these two cases, we also show that the outer bound is within half bits of the capacity region for Gaussian BCs with arbitrary noise correlation. Lastly, it is shown that for the Gaussian BC, regardless of the capacities of conferencing links, a one-sided cooperation scheme (from the stronger user to the weaker one) based on decode-and-forward is sufficient to achieve the capacity region to within $\frac{1}{2}\log(\frac{2}{1-|\lambda|})$ bits, where $\lambda$ is the correlation coefficient of the receiver noises.

## II. Preliminaries

The two-user BC with conferencing decoders is a communication scenario in which a transmitter sends a common message and two private messages to the two users. The receivers are able to exchange information via communication links of finite capacities called conferencing links. Fig. 1 illustrates the channel model. The channel is given by $(\mathcal{X}, \mathcal{Y}_1, \mathcal{Y}_2, P(y_1, y_2|x), C_{12}, C_{21})$ where $\mathcal{X}$ denotes input alphabet, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ denote output alphabets, $P(y_1, y_2|x)$ is the channel transition probability function, and $C_{12}$, and $C_{21}$ are capacities of the conferencing links.

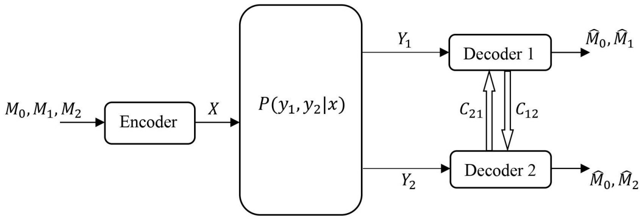


Figure 1. The two-user BC with conferencing decoders.

This paper focuses on the Gaussian BC with correlated noises defined as follows:

$$\begin{cases} Y_1 = aX + Z_1 \\ Y_2 = bX + Z_2, \end{cases} \tag{1}$$

where $Z_1$ and $Z_2$ are correlated Gaussian random variables with zero mean and unit variances and correlation coefficient $\lambda$, i.e., $\mathbb{E}[Z_1 Z_2] = \lambda$, $X$ is the input signal with $\mathbb{E}[X^2] \leq P$, and $a$ and $b$ are real-valued channel gains. When there is no loss of generality, we assume that $|a| \geq |b|$, where $|\cdot|$ stands for absolute value.

The definitions of encoding, decoding processes, and the definition of capacity region of the two-user BC with conferencing decoders can be found in [8], which is essentially that of [2]. Throughout the paper, we use the notation $\psi(x) \triangleq \frac{1}{2}\log(1 + x)$.

## III. Main Results

### A. Converse

First of all, recall the converse theorem in the previous work [8] for the general BC with conferencing decoders.

**Theorem 1.** *( [8]) Consider the two-user BC with conferencing decoders shown in Fig. 1. Let $\mathcal{R}_o$ denote the set of all rate triples $(R_0, R_1, R_2)$ such that:*

$$R_0 + R_1 \leq I(U; Y_1) + C_{21} \tag{2}$$
$$R_1 \leq I(X; Y_1|Y_2, V) + I(X; Y_2) \tag{3}$$
$$R_1 \leq I(X; Y_2|Y_1, V) + I(X; Y_1) \tag{4}$$
$$R_0 + R_2 \leq I(V; Y_2) + C_{12} \tag{5}$$
$$R_2 \leq I(X; Y_2|Y_1, U) + I(X; Y_1) \tag{6}$$
$$R_2 \leq I(X; Y_1|Y_2, U) + I(X; Y_2) \tag{7}$$
$$R_0 + R_1 + R_2 \leq I(X; Y_1|V) + I(V; Y_2) + C_{12} + C_{21} \tag{8}$$
$$R_0 + R_1 + R_2 \leq I(X; Y_2|U) + I(U; Y_1) + C_{12} + C_{21} \tag{9}$$
$$R_0 + R_1 + R_2 \leq I(X; Y_1|Y_2, V) + I(X; Y_2) + C_{12} \tag{10}$$
$$R_0 + R_1 + R_2 \leq I(X; Y_2|Y_1, U) + I(X; Y_1) + C_{21} \tag{11}$$
$$R_0 + R_1 + R_2 \leq I(X; Y_1, Y_2) \tag{12}$$

*for some joint PDFs $P_{U,V,X}$ where $U, V \to X \to Y_1, Y_2$ forms a Markov chain. The set $\mathcal{R}_o$ constitutes an outer bound on the capacity region.*

A complete proof of Theorem 1 is given in [8]. It is based on multiple applications of the Csiszár-Körner identity. In this paper, we focus on the Gaussian BC (1). By considering the input power constraint $\mathbb{E}[X^2] \leq P$, we can optimize the bound over its auxiliary variables $U$ and $V$ for the Gaussian channel and derive an explicit characterization of the mutual information terms in the outer bound as below.

**Theorem 2.** *Consider the Gaussian BC (1) with conferencing decoders. Assume that $|a| \geq |b|$. Let $\mathcal{R}_o^G$ denote the set of all rate triples $(R_0, R_1, R_2)$ such that for some $\alpha, \beta \in [0, 1]$:*

$$R_0 + R_1 \leq \psi\left(\frac{(1-\alpha)a^2 P}{\alpha a^2 P + 1}\right) + C_{21} \tag{13}$$
$$R_1 \leq \Psi_2 + \psi\left(\frac{(1-\beta)b^2 P}{\beta b^2 P + 1}\right) \tag{14}$$
$$R_0 + R_2 \leq \psi\left(\frac{(1-\beta)b^2 P}{\beta b^2 P + 1}\right) + C_{12} \tag{15}$$
$$R_2 \leq \Psi_1 + \psi\left(\frac{(1-\alpha)b^2 P}{\alpha b^2 P + 1}\right) \tag{16}$$
$$R_0 + R_1 + R_2 \leq \psi\left(\beta a^2 P\right) + \psi\left(\frac{(1-\beta)b^2 P}{\beta b^2 P + 1}\right) + C_{12} + C_{21} \tag{17}$$
$$R_0 + R_1 + R_2 \leq \Psi_2 + \psi\left(\frac{(1-\beta)b^2 P}{\beta b^2 P + 1}\right) + C_{12} \tag{18}$$
$$R_0 + R_1 + R_2 \leq \Psi_1 + \psi\left(\frac{(1-\alpha)a^2 P}{\alpha a^2 P + 1}\right) + C_{21} \tag{19}$$
$$R_0 + R_1 + R_2 \leq \psi\left(\left(\frac{a^2 + b^2 - 2\lambda ab}{1 - \lambda^2}\right) P\right), \tag{20}$$

*where*

$$\Psi_1 = \psi\left(\alpha\left(\frac{a^2 + b^2 - 2\lambda ab}{1 - \lambda^2}\right) P\right)$$
$$\Psi_2 = \psi\left(\beta\left(\frac{a^2 + b^2 - 2\lambda ab}{1 - \lambda^2}\right) P\right).$$

*The set $\mathcal{R}_o^G$ constitutes an outer bound on the capacity region.*

The outer bound is based on applying entropy power inequality on the outer bound derived in Theorem 1. The details are omitted due to space limitation.

**Remark 1.** *For the Gaussian BC (1) if $\lambda = \frac{b}{a}$, the channel is degraded and we have:*

$$\frac{a^2 + b^2 - 2\lambda ab}{1 - \lambda^2} = a^2 + \frac{(\lambda a - b)^2}{1 - \lambda^2} = a^2 \tag{21}$$

*In this case, the outer bound $\mathcal{R}_o^G$ can be shown to be achievable, and it yields the capacity region (see Remark 5).*

Thus, for the rest of the paper, we assume that $\lambda \neq \frac{b}{a}$.

### B. Achievable Rate Region

In [2, Theorem 2] and [4, App. B], achievable rate regions are given for the BC with conferencing links. However, these regions (which are given for the channel with private messages only) are in general insufficient for deriving new capacity results. The achievable rate region below is for the two-user BC with both common and private messages and bidirectional conferencing receivers. In the proposed coding scheme, we apply Marton's coding as the transmission scheme, and quantize-bin-and-froward at one receiver and decode-and-froward at the other receiver as the cooperative strategy. This result is from [8], and when specialized to the Gaussian case, it can give capacity results in several cases.

**Theorem 3.** *( [8]) Consider the two-user BC with conferencing decoders shown in Fig. 1. Let $\mathcal{R}_i^{(1)}$ denote the set of all rate triples $(R_0, R_1, R_2)$ such that*

$$\begin{aligned} R_0 + R_1 &\leq \min\left\{I(U,W;Y_1)+\zeta, I(U,W;Y_1,\hat{Y}_2)\right\} \\ R_0 + R_2 &\leq I(V,W;Y_2)+C_{12} \\ R_0+R_1+R_2 &\leq \min\left\{I(U;Y_1|W)+\zeta, I(U;Y_1,\hat{Y}_2|W)\right\} \\ &\quad + I(V,W;Y_2)+C_{12}-I(U;V|W) \\ R_0+R_1+R_2 &\leq \min\left\{I(U,W;Y_1)+\zeta, I(U,W;Y_1,\hat{Y}_2)\right\} \\ &\quad + I(V;Y_2|W)-I(U;V|W) \\ 2R_0+R_1+R_2 &\leq \min\left\{I(U,W;Y_1)+\zeta, I(U,W;Y_1,\hat{Y}_2)\right\} \\ &\quad + I(V,W;Y_2)+C_{12}-I(U;V|W) \\ \zeta &= \{C_{21}-I(\hat{Y}_2;Y_2|W,U,Y_1)\}^+ \end{aligned} \tag{22}$$

*for some joint PDFs $P(u,v,w,x)P(y_1,y_2|x)P(\hat{y}_2|y_2)$. The convex closure of the set $\mathcal{R}_i^{(1)}$ is achievable.*

It is clear that a second achievable rate region can be derived by exchanging the order of cooperation at the receivers. Moreover, one may consider applying multiple rounds of cooperation. However, we demonstrate that the region (22) with one round is already sufficient to prove several new capacity (and approximate capacity) results for the Gaussian BC with conferencing receivers and with correlated noises.

**Theorem 4.** *Consider the two-user Gaussian BC (1) with degraded message sets (i.e., a common message for both receivers and a private message for the first receiver) and bidirectional conferencing receivers. Assume $\lambda \neq \frac{b}{a}$. For the channel with fully correlated noises where $|\lambda| = 1$, the capacity region is given by:*

$$R_0 \leq \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P+1}\right)+C_{12} \tag{23}$$

$$R_0+R_1 \leq \psi\left(a^2P\right)+C_{21} \tag{24}$$

$$R_0+R_1 \leq \psi\left(\beta a^2 P\right)+\psi\left(\frac{(1-\beta)b^2P}{\beta b^2P+1}\right)+C_{12}+C_{21} \tag{25}$$

*for some $\beta \in [0,1]$.*

*Proof:* The achievability is derived from $\mathcal{R}_i^{(1)}$ given in Theorem 3 by setting $X \equiv U \equiv W + \bar{W}$ and $V \equiv \varnothing$, and $\hat{Y}_2 \equiv Y_2 + \hat{Z}_2$, where $W$ and $\bar{W}$ are two independent Gaussian variables with zero means and variances $(1-\beta)P$ and $\beta P$, respectively, and $\hat{Z}_2$ is a Gaussian variable (independent of all other variables) with zero mean and variance $\hat{\sigma}^2$. Note that by this choice of variables, when $|\lambda| = 1$, we have:

$$I(\hat{Y}_2;Y_2|W,U,Y_1) = \frac{1}{2}\psi\left(\frac{1-\lambda^2}{\hat{\sigma}^2}\right)=0$$

Moreover,

$$\begin{aligned} I(X;Y_1,\hat{Y}_2) &= \frac{1}{2}\psi\left(\left(\frac{a^2+b^2-2\lambda ab+\hat{\sigma}^2a^2}{1-\lambda^2+\hat{\sigma}^2}\right)P\right) \\ &= \frac{1}{2}\psi\left(\left(\frac{a^2+b^2-2\lambda ab+\hat{\sigma}^2a^2}{\hat{\sigma}^2}\right)P\right) \end{aligned}$$

and

$$\begin{aligned} I(X;Y_1,\hat{Y}_2|W) &= \frac{1}{2}\psi\left(\beta\left(\frac{a^2+b^2-2\lambda ab+\hat{\sigma}^2a^2}{1-\lambda^2+\hat{\sigma}^2}\right)P\right) \\ &= \frac{1}{2}\psi\left(\beta\left(\frac{a^2+b^2-2\lambda ab+\hat{\sigma}^2a^2}{\hat{\sigma}^2}\right)P\right). \end{aligned}$$

Therefore, by letting $\hat{\sigma}^2 \to 0$, one can make the above two mutual information terms arbitrary large and thus they would not be in effect in the characterization of the region (22). For the case of $|a| \geq |b|$, the converse proof is readily given by $\mathcal{R}_o^G$ in Theorem 2. Note that the constraint (13) is stricter than (24). For the case of $|a| < |b|$, the rate region (23)-(25) (which is still achievable by the proposed scheme) is optimal for $\beta = 0$ and thereby it coincides with the cut-set bound. ■

**Remark 2.** *The capacity characterization given in Theorem 4 gives the following interesting observation. Even with a very small (yet positive) amount of input power $P$, one can transmit information over the channel at a rate as high as the capacities of the conferencing links, if the noises are fully correlated. In fact, if $P \to 0$, the capacity region is as follows:*

$$\begin{aligned} R_0 &\leq \epsilon_1(P)+C_{12} \\ R_0+R_1 &\leq \epsilon_2(P)+C_{21}, \end{aligned} \tag{26}$$

*for some $\epsilon_1(P)$ and $\epsilon_2(P)$, which go to zero as as $P \to 0$.*

Note that the argument presented in the proof of Theorem 4 is only valid for strictly positive values of input power $P$.

**Remark 3.** *For the special case of the relay channel, i.e., $Y_2$ acts as a relay for $Y_1$, $C_{12} = 0$ and there is only a private message for the first receiver, the capacity result given in Theorem 4 is reduced to $R_1 \leq \psi\left(a^2P\right)+C_{21}$. In this case, using the quantize-bin-forward strategy, one can achieve the cut-set bound. Interestingly, the capacity does not depend on $b$ at all. This result is an example of a semi-deterministic primitive relay channel, because the relay observation $Y_2 = bX + Z_2$ is a deterministic function of input $X$ and the receiver observation $Y_1 = aX + Z_1$ when $Z_1$ and $Z_2$ are fully correlated. In this case, the cut-set bound is achievable [11].*

**Example 1.** A special case of the Gaussian BC with $|\lambda| = 1$ is the following scenario:

$$\begin{cases} Y_1 = X+Z \\ Y_2 = X-Z, \end{cases} \tag{27}$$

where $Z$ is a zero mean unit variance Gaussian noise. In this scenario, the two receivers see exactly the same noise but with a different sign. For this channel, the capacity result of Theorem 4 reduces to the following:

$$\begin{aligned} R_0 &\leq \psi(P) + C_{12} \\ R_0 + R_1 &\leq \psi(P) + C_{21}. \end{aligned} \tag{28}$$

In fact for this channel, the cut-set bound is achievable. To achieve this capacity, the second receiver applies quantize-bin-and-forward and the first receiver applies decode-and-forwards as the cooperation protocol. As already mentioned, for the special case of $R_0 = 0$ and $C_{12} = 0$, this result recovers the capacity of semi-deterministic relay channel [11, Example 1].

**Example 2.** Consider the following Gaussian channel:

$$\begin{cases} Y_1 = X + Z \\ Y_2 = Z, \end{cases} \tag{29}$$

where $Z$ is a zero mean unit variance Gaussian noise. In this case, the user $Y_2$ does not receive any information from the transmitter and only observes the additive noise of the user $Y_1$. For this channel, the capacity region can be derived by setting $a = 1$ and $b = 0$ in (23)-(25) and is given by:

$$\begin{aligned} R_0 &\leq C_{12} \\ R_0 + R_1 &\leq \psi(P) + C_{21}. \end{aligned} \tag{30}$$

To achieve this capacity, first the user $Y_2$ sends a compressed version of the observed noise $Z$ to the user $Y_1$ through the digital link $C_{21}$. Next, the user $Y_1$ decodes both common and private messages based on the message from $Y_2$ then forwards the common message to $Y_2$ through the digital link $C_{12}$.

**Example 3.** Consider the following Gaussian channel:

$$\begin{cases} Y_1 = Z \\ Y_2 = X + Z. \end{cases} \tag{31}$$

In this case, the capacity region is given by:

$$\begin{aligned} R_0 &\leq \psi(P) + C_{12} \\ R_0 + R_1 &\leq C_{21}. \end{aligned} \tag{32}$$

Note that in this channel, the user $Y_1$ (which is supposed to detect both common and private messages) does not receive any information from the transmitter directly. The capacity achieving cooperation protocol is that the user $Y_2$ first sends a compressed version of its received signal to the user $Y_1$ through the digital link $C_{21}$. Next, the user $Y_1$ decodes both messages using the information received (and its own signal, which is in fact the channel noise) then forwards part of the common message to the user $Y_2$ through the link $C_{12}$. Lastly, the user $Y_2$ decodes the unknown part of the common message using its received signal.

Note that in Example 3, as the user $Y_1$ observes the channel noise only, one would think that a cooperative scheme in which $Y_1$ applies compress-and-forward and $Y_2$ applies decode-and-forward should be used. However, such a scheme is not optimal in this particular BC with degraded message for $Y_2$.

The capacity result in Theorem 4 is important from two viewpoints. First, it is the first capacity result for a two-user Gaussian BC with bidirectional cooperation between receivers (all previously capacity results are regarding one-sided cooperation). Second, this result is among the rare cases in network information theory for which quantize-bin-and-forwards strategy contributes to achieving capacity.

The previous capacity result is about BC with degraded message sets. In the next theorem, for the Gaussian BC with one-sided cooperation and fully correlated noises, we establish the exact capacity region of the channel with both common and private messages.

**Theorem 5.** *Consider the two-user Gaussian BC (1) with both common and private messages where $|a| \geq |b|$ and only the weaker receiver is connected to the stronger one by a conferencing link, i.e., $C_{12} = 0$. Assume $\lambda \neq \frac{b}{a}$. For the channel with fully correlated noises where $|\lambda| = 1$, the capacity region is given by:*

$$\begin{aligned} R_0 + R_2 &\leq \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) \\ R_0 + R_1 + R_2 &\leq \psi\left(\beta a^2 P\right) + \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) + C_{21} \end{aligned} \tag{33}$$

*for some $\beta \in [0, 1]$.*

The proof of Theorem 5 is similar to that of Theorem 4 and therefore is omitted here.

For Gaussian BCs in which the noises are not fully correlated, the inner and outer bounds of this paper yield approximate capacity results which are presented in the following theorems. The first approximate capacity result is on Gaussian BC with degraded message set.

**Theorem 6.** *Consider the two-user Gaussian BC (1) with degraded message sets (i.e., a common message for both receivers and a private message for the first receiver) and bidirectional conferencing receivers. Assume that $|a| \geq |b|$ and $\lambda \neq \frac{b}{a}$. For all channel parameters a, b, $C_{12}$, $C_{21}$, and $\lambda$ with $|\lambda| < 1$, the following achievable rate region is within half bits to the capacity region:*

$$R_0 \leq \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) + C_{12} \tag{34}$$

$$R_0 + R_1 \leq \psi\left(a^2 P\right) + \{C_{21} - 1/2\}^+ \tag{35}$$

$$R_0 + R_1 \leq \psi\left(\left(\frac{a^2 + b^2 - 2\lambda ab + (1-\lambda^2)a^2}{2(1-\lambda^2)}\right) P\right) \tag{36}$$

$$\begin{aligned} R_0 + R_1 \leq{}& \psi\left(\beta a^2 P\right) + \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) \\ &+ \{C_{21} - 1/2\}^+ + C_{12} \end{aligned} \tag{37}$$

$$\begin{aligned} R_0 + R_1 \leq{}& \psi\left(\beta\left(\frac{a^2 + b^2 - 2\lambda ab + (1-\lambda^2)a^2}{2(1-\lambda^2)}\right) P\right) \\ &+ \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) + C_{12} \end{aligned} \tag{38}$$

*for some $\beta \in [0, 1]$.*

*Proof:* The above achievable rate region is derived from $\mathcal{R}_i^{(1)}$ given in Theorem 3 by setting $X \equiv U \equiv W + \bar{W}$ and $V \equiv \varnothing$, and $\hat{Y}_2 \equiv Y_2 + \hat{Z}_2$, where $W$ and $\bar{W}$ are two independent Gaussian variables with zero means and variances $(1-\beta)P$ and $\beta P$, respectively, and $\hat{Z}_2$ is a Gaussian variable (independent of all other variables) with zero mean and variance $\hat{\sigma}^2 = 1 - \lambda^2$.

By a simple comparison, one can see that the right-hand sides of the constraints (34), (35), (36), (37) and (38) are within half bits of (15), (13), (20), (17), and (18), respectively. ■

Next, we present an approximate capacity result for the Gaussian BC with one-way conferencing.

**Theorem 7.** *Consider the two-user Gaussian BC (1) with both common and private messages, where $|a| \geq |b|$ and only the weaker receiver has a conferencing link to the stronger receiver, i.e., $C_{12} = 0$. Assume $\lambda \neq \frac{b}{a}$. For all channel parameters $a$, $b$, $C_{21}$, and $\lambda$ with $|\lambda| < 1$, the following achievable rate region is within half bits to the capacity region:*

$$R_0 + R_2 \leq \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) \tag{39}$$

$$R_0 + R_1 + R_2 \leq \psi\left(\left(\frac{a^2 + b^2 - 2\lambda ab + (1-\lambda^2)a^2}{2(1-\lambda^2)}\right)P\right) \tag{40}$$

$$R_0 + R_1 + R_2 \leq \psi\left(\beta a^2 P\right) + \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) + \{C_{21} - 1/2\}^+ \tag{41}$$

$$R_0 + R_1 + R_2 \leq \psi\left(\beta\left(\frac{a^2 + b^2 - 2\lambda ab + (1-\lambda^2)a^2}{2(1-\lambda^2)}\right)P\right) + \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) \tag{42}$$

*for some $\beta \in [0, 1]$.*

*Proof:* Similar to Theorem 6, the above achievable rate region is derived from $\mathcal{R}_i^{(1)}$ given in Theorem 3 by setting $X \equiv U \equiv W + \bar{W}$ and $V \equiv \varnothing$, and $\hat{Y}_2 \equiv Y_2 + \hat{Z}_2$, where $W$ and $\bar{W}$ are two independent Gaussian variables with zero means and variances $(1-\beta)P$ and $\beta P$, respectively, and $\hat{Z}_2$ is a Gaussian variable (independent of all other variables) with zero mean and variance $\hat{\sigma}^2 = 1 - \lambda^2$.

By a simple comparison, one can see that the right-hand sides of the constraints (39), (40), (41) and (42) are within half bits of (15), (20), (17), and (18), respectively. ■

Finally, we derive an approximate capacity result for the two-user Gaussian BC (1) with both common and private messages and with bidirectional cooperative receivers. First, we present an achievable region for the channel using only one-way conferencing with decode-and-forward. The other conferencing link is not used, so the resulting achievable rate region is a sub-region of $\mathcal{R}_i^{(1)}$ (22). It turns out that this region is already approximately optimal when the noise correlation is small.

**Corollary 1.** *Let $\mathcal{R}_i^{DF-G}$ denote the set of all rate triples $(R_0, R_1, R_2)$ such that*

$$R_0 + R_2 \leq \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) + C_{12} \tag{43}$$

$$R_0 + R_1 + R_2 \leq \psi\left(a^2 P\right) \tag{44}$$

$$R_0 + R_1 + R_2 \leq \psi\left(\beta a^2 P\right) + \psi\left(\frac{(1-\beta)b^2P}{\beta b^2 P + 1}\right) + C_{12} \tag{45}$$

*for some $\beta \in [0, 1]$. The set $\mathcal{R}_i^{DF-G}$ constitutes an inner bound on the capacity region of the Gaussian BC (1) with conferencing decoders.*

*Proof:* The bound $\mathcal{R}_i^{DF-G}$ is derived from $\mathcal{R}_i^{(1)}$ given in Theorem 3 by setting $X \equiv U \equiv W + \bar{W}$ and $V \equiv \hat{Y}_2 \equiv \varnothing$, where $W$ and $\bar{W}$ are two independent Gaussian variables with zero means and variances $\beta P$ and $(1-\beta)P$, respectively. Note that this inner bound is in fact derived for the channel with one-sided cooperation (i.e., it does not make use of the conferencing link $C_{21}$) using the decode-and-forward strategy alone. Nevertheless, it is a valid inner bound for the channel with bidirectional cooperation. ■

**Theorem 8.** *Consider the two-user Gaussian BC (1) with both common and private messages and with bidirectional conferencing decoders. Assume that $|a| \geq |b|$. For all channel parameters $a$, $b$, $C_{12}$, $C_{21}$, and $\lambda$, the inner bound $\mathcal{R}_i^{DF-G}$ is within $\frac{1}{2}\log(\frac{2}{1-|\lambda|})$ bits of the capacity region.*

*Proof:* The constraint (43) is identical to (15). Moreover, by simple algebraic computations, one can show that the right-hand sides of the constraints (44) and (45) are within $\frac{1}{2}\log(\frac{2}{1-|\lambda|})$ bits of (20) and (18), respectively. ■

**Remark 4.** *For Gaussian BCs with $\lambda ab \geq 0$, a better approximate capacity bound of $\frac{1}{2}\log(\frac{2}{1-\lambda^2})$ bits is possible for the region $\mathcal{R}_i^{DF-G}$ given in Corollary 1.*

Theorem 8 states that if we do not use the quantize-bin-forward part of the conferencing protocol and rely solely on decode-and-forward, the gap to capacity would depend on the noise correlation. This is because decode-and-forward cannot exploit the noise correlation, so while it achieves within constant gap to the capacity region when the noises are uncorrelated, it cannot do so when the noises are highly correlated. Nevertheless, there is one special case for which decode-and-forward is optimal.

**Remark 5.** *For the Gaussian BC (1) with $\lambda = \frac{b}{a}$, considering (21), one can verify that the decode-and-forward achievable region $\mathcal{R}_i^{DF-G}$ of Corollary 1 coincides with the outer bound $\mathcal{R}_o^G$ given in Theorem 2. Thus, it yields the capacity region.*

As concluding remark, the novel structure of the outer bound given in Theorem 2, in particular, the constraint (18), is crucial for deriving the exact capacity results in Theorems 4 and 5, and also the approximate capacity results in Theorems 6, 7, and 8.

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