Bounds on the Minimum Number of Beamformers for Integrated Sensing and Communications

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***Abstract*—Consider a downlink integrated sensing and communications system where a base station employs linear beamforming to estimate an unknown vector of L real parameters, while communicating with K users. What is the minimum number of beamformers needed to simultaneously perform both tasks? This paper first shows that the minimum number of beamforming vectors is bounded by $K + \sqrt{L(L+1)/2}$, when the sensing performance is measured in terms of the Cramér-Rao bound involving an $L \times L$ Fisher information matrix, and the communications performance is measured in terms of signal-to-noise-and-interference ratios. This bound can be tightened and also generalized by recognizing that when the sensing metric is a function of M quadratic terms involving the beamformers, the minimum number of beamformers is at most $K + \sqrt{M}$, where M can be less than $L(L+1)/2$. In particular, for the task of estimating the complex path loss and the angle-of-arrival (AoA) of N targets (with a total of $L = 3N$ parameters), due to the interdependencies in estimating these parameters through the beamformers, we show that the total number of beamformers needed is at most $K + \sqrt{3.5N^2 + 0.5N}$. For the sensing-only scenario with $K = 0$, the minimum number of beamformers needed is asymptotically bounded by $1.871N$.**

I. Introduction

Wireless communications systems and radars systems are historically developed independently. In recent years, integration of sensing and communication (ISAC) has emerged as an alternative to the traditional approach of operating sensing and communication systems separately [1]. By combining the functionalities of communications and radar, ISAC offers the potential for greater spectrum efficiency, improved hardware utilization, and reduced overall cost.

Synthesizing transmit waveforms that simultaneously carry information and enable radar-like functions is a central idea in ISAC system design [2]–[5]. In general, an ISAC waveform is formed by superimposing a radar signal on top of a communication signal. In the context of MIMO beamforming, this translates to adding extra beamformers for sensing [2], [3] alongside the communication beamformers. Most current ISAC studies simply assume that the total number of beamformers should be maximally set to be the number of antennas. While the associated optimization problem becomes easily tractable via semidefinite relaxation (SDR) with such a design choice, it may also unnecessarily complicate the system implementation (because the same performance can be attained using fewer beamformers, e.g., see [6]).

This paper explores the question of what is the minimum number of beamformers needed for simultaneous communications and sensing. An exact answer to this question is difficult to obtain, because: i) sensing and communication beamformers are interdependent and can affect each other's performance; and ii) the answer depends on the specific estimation task since there is no universal metric applicable to all estimation problems. For this reason, even the answer to this question for the seemingly simple case of MIMO radar without communication users is already highly nontrivial.

In the literature, some partial characterizations of the minimum number of beamformers needed for sensing are available in a handful of special cases. In [7], it is shown that the number of beamformers for angle-of-arrival (AoA) estimation in MIMO radar (without communication users) is at most twice the number of targets. This bound is established based on using the classical Cramér-Rao bound (CRB) as the sensing performance metric, which requires the unrealistic assumption that the AoA to be estimated is known ahead of time. Recent works [4], [6] examine the ISAC framework. Specifically, [4] shows that two beamformers are needed for the simple case of one target and one user when the Bayesian CRB (BCRB) is adopted. In the meanwhile, [6] considers a detection problem and adopts signal-to-noise-ratio (SNR) to measure the detection performance. It is shown that no additional beamformers are needed (i.e., beyond the communication ones) when there is one target and any number of communications users.

This work derives novel bounds on the minimum number of beamformers needed to simultaneously communicate to any number of users and to estimate any number of parameters, based on BCRB. We show that for estimating L parameters and communicating with K users, the minimum number of beamformers cannot exceed $K + \sqrt{L(L+1)/2}$. Furthermore, in an ISAC system with K communication users and for estimating the complex path losses and the AoAs of N line-of-sight (LoS) targets (with $L = 3N$ parameters), the minimum number of beamformers is at most $K + \sqrt{3.5N^2 + 0.5N}$. Setting $K = 0$ for the sensing-only case, this implies that the minimum number of beamformers needed is two for $N = 1$, three for $N = 2$, and asymptotically bounded by $1.871N$ for large N. This is tighter than the $2N$ bound derived in [7] under the classical CRB. When comparing to the special case of $K = N = 1$ as in [4] and the case of $N = 1$ with SNR as the detection metric as in [6], the bounds in the paper are slightly looser by one beamformer. However, the bounds derived in this paper are applicable more generally for arbitrary K and N, rather than just for the special cases studied in [4] and [6].

II. Downlink Beamforming for ISAC

A. System Model

Consider a downlink ISAC system where a base station (BS) with $N_{\rm T}$ transmit antennas aims to send information to K single-antenna users and to simultaneously learn a vector of L real-valued parameters $\boldsymbol{\eta} \in \mathbb{R}^L$ by listening to the reflected signal through its $N_{\rm R}$ received antennas in full-duplex mode. The input-output relationships at the remote communication users and at the full-duplex received antennas at the BS for sensing in each symbol period are described by

$$y_k = \mathbf{h}_k^{\mathsf{H}}\mathbf{x} + z_{\rm k}^{\mathsf{H}}, \quad \forall k \in \{1, \dots, K\}, \tag{1a}$$

$$\mathbf{y}_{\rm s} = \mathbf{G}^{(\boldsymbol{\eta})}\mathbf{x} + \mathbf{z}_{\rm s}, \tag{1b}$$

where $\mathbf{x} \in \mathbb{C}^{N_{\rm T}}$ is the transmitted signal, $\mathbf{h}_k \in \mathbb{C}^{N_{\rm T}}$ is the channel vector, and $y_K \in \mathbb{C}^T$ is the received signal for the k-th communication user. The matrix $\mathbf{G}^{(\boldsymbol{\eta})} \in \mathbb{C}^{N_{\rm R}\times N_{\rm T}}$ models the "round-trip" channel between the BS transmit and receive arrays, and $\mathbf{y}_{\rm s} \in \mathbb{C}^{N_{\rm R}}$ is the received radar signal at the BS. We make the simplifying assumption that $\mathbf{G}^{(\boldsymbol{\eta})}$ is a deterministic function of the parameters $\boldsymbol{\eta}$. Finally, $z_k \in \mathbb{C}$ and $\mathbf{z}_{\rm s} \in \mathbb{C}^{N_{\rm R}}$ are Gaussian noises with i.i.d. entries $\sim \mathcal{CN}(0, \sigma^2)$.

We assume a block fading model for both the communication and sensing channels, where $\mathbf{h}_1, \dots, \mathbf{h}_k$ and $\boldsymbol{\eta}$ remain fixed during a coherence interval T. Within the coherence interval, we assume that the communication channel $\mathbf{H} \triangleq [\mathbf{h}_1, \dots, \mathbf{h}_K]$ can be perfectly estimated by using pilots. The vector $\boldsymbol{\eta}$ is unknown and to be estimated. We adopt a Bayesian framework and assume that it has a prior distribution $f_{\boldsymbol{\eta}}(\boldsymbol{\eta})$.

We adopt a linear beamforming scheme for both communications and sensing, where the transmit signal is formed as the sum of a communication part and a sensing part. This is done by augmenting the "traditional" beamforming model for communications with extra beamformers dedicated to sensing [2], [3]. In particular, the t-th transmit signal within the coherence interval of length T is formed by

$$\mathbf{x}^{(t)} \triangleq \begin{bmatrix}\mathbf{V}_{\rm c} & \mathbf{V}_{\rm s}\end{bmatrix}\begin{bmatrix}\mathbf{s}_{\rm c}^{(t)} \\ \mathbf{s}_{\rm s}^{(t)}\end{bmatrix}, \tag{2}$$

where $\mathbf{V}_{\rm c} \triangleq [\mathbf{v}_1, \dots, \mathbf{v}_K] \in \mathbb{C}^{N_{\rm T}\times K}$ is the set of communication beamformers with $\mathbf{v}_k$ denoting the beamformer for the k-th user, and $\mathbf{V}_{\rm s} \in \mathbb{C}^{N_{\rm T}\times(N-K)}$ is the matrix of additional sensing beamformers. Note that extra $(N-K)$ beamformers are added for sensing, so the total number of beamformers is N. Here, $\mathbf{s}_{\rm c}^{(t)} \in \mathbb{C}^K$ are the communication symbols and $\mathbf{s}_{\rm s}^{(t)} \in \mathbb{C}^{N-K}$ are the pseudo-random sequences for sensing, both assumed to have i.i.d. $\mathcal{CN}(0,1)$ entries, over $t = 1\cdots, T$. The overall beamforming matrix $\mathbf{V} \triangleq \begin{bmatrix}\mathbf{V}_{\rm c} & \mathbf{V}_{\rm s}\end{bmatrix}$ needs to satisfy a total power constraint $\operatorname{Tr}\left(\mathbf{V}\mathbf{V}^{\mathsf{H}}\right) \le P$.

B. Performance Metrics

The communication performance is measured in terms of the achievable rate for each user, which is a function of the signal-to-interference-and-noise ratio (SINR) at the receiver

$$\mathrm{SINR}_k^{(\mathbf{V})} \triangleq \frac{|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_k|^2}{\sum_{i\neq k}|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}_i|^2 + \mathbf{h}_k^{\mathsf{H}}\mathbf{V}_{\rm s}\mathbf{V}_{\rm s}^{\mathsf{H}}\mathbf{h}_k + \sigma^2}. \tag{3}$$

The previous SINR expression arises from treating the radar signal as interference. In theory, the interference from the beamformed sensing signal can be canceled, but we do not assume cancellation here in order to simplify transciever design, as commonly done in the ISAC literature [2]–[4].

The sensing performance is measured in terms of bounds on the mean-squared-error (MSE) for estimating the parameters of interest. In this paper, we assume a prior $f_{\boldsymbol{\eta}}(\cdot)$, and use the BCRB as a lower bound on the MSE averaged over $f_{\boldsymbol{\eta}}(\cdot)$ [8]:

$$\mathbb{E}_{\boldsymbol{\eta}}\left[\mathbb{E}\left[(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})(\boldsymbol{\eta}-\hat{\boldsymbol{\eta}})^T\,\middle|\,\boldsymbol{\eta}\right]\right] \succcurlyeq (\mathbf{J}^{(\mathbf{V})})^{-1} \tag{4}$$

where $\succcurlyeq$ denotes inequality with respect to the positive semidefinite (PSD) cone and $\hat{\boldsymbol{\eta}}$ is an estimate of $\boldsymbol{\eta}$ satisfying certain regularity conditions. The matrix $\mathbf{J}^{(\mathbf{V})}$ denotes the $L\times L$ Bayesian Fisher Information matrix (BFIM), which can be expressed by

$$\mathbf{J}^{(\mathbf{V})} = \mathbf{C} + \mathbf{T}^{(\mathbf{V})}, \tag{5}$$

where $\mathbf{C}$ is a matrix that depends on the prior only, and $\mathbf{T}^{(\mathbf{V})}$ is a PSD matrix whose (i,j)-th element is given by [5]

$$[\mathbf{T}_{\mathbf{V}}]_{ij} \triangleq \frac{T}{\sigma^2}\operatorname{Tr}\left(\tilde{\mathbf{G}}_{ij}\mathbf{V}\mathbf{V}^{\mathsf{H}}\right) \tag{6}$$

with $\tilde{\mathbf{G}}_{ij} \triangleq \mathbb{E}\left[\dot{\mathbf{G}}_i^{\mathsf{H}}\dot{\mathbf{G}}_j + \dot{\mathbf{G}}_j^{\mathsf{H}}\dot{\mathbf{G}}_i\right]$ and $\dot{\mathbf{G}}_i \triangleq \frac{\partial \mathbf{G}^{(\boldsymbol{\eta})}}{\partial \eta_i}$. Note that the BCRB is a function of *both* communications beamformers and the extra sensing beamformers, because the echoes from both the communication symbols and the sensing sequences are used for the estimation of $\boldsymbol{\eta}$.

C. The Minimum Number of Beamformers

The total number of beamforming vectors for the ISAC operation should clearly satisfy $K \le N \le N_{\rm T}$, because at least K beamformers are needed to communicate to the K users, and at most $N_{\rm T}$ beamformers can be used since there are a total of $N_{\rm T}$ antennas. However, setting N to either K or $N_{\rm T}$ is undesirable. Setting $N = K$ may not give us sufficient transmit dimensions to achieve the best sensing performance; setting $N = N_{\rm T}$ may be unnecessary since we can often achieve the same performance using fewer beamformers without overcomplicating the system implementation.

The goal of this paper is to characterize the minimum number of beamformers needed for ISAC such that it results in no performance loss relative to $N = N_{\rm T}$, i.e.,

$$N_{\rm min}^{\rm BCRB} \triangleq \arg\min \quad N \tag{7a}$$

$$\text{subject to} \quad p_N^* = p_{N_{\rm T}}^*, \tag{7b}$$

where p_N^* is the optimal value of an ISAC beamforming problem with SINR targets γ_k and power constraint P, assuming that N beamformer vectors are used, i.e.,

$$\underset{\mathbf{V}\in\mathbb{C}^{N_{\rm T}\times N}}{\text{minimize}} \quad h\left(\left(\mathbf{J}^{(\mathbf{V})}\right)^{-1}\right) \tag{8a}$$

$$\text{subject to} \quad \mathrm{SINR}^{(\mathbf{V})} \ge \gamma_k, \quad \forall k. \tag{8b}$$

$$\operatorname{Tr}(\mathbf{V}\mathbf{V}^{\mathsf{H}}) \le P. \tag{8c}$$

Here, $h(\cdot)$ is a nondecreasing scalar function of the BCRB. For instance, trace, weighted-trace, or logarithm-determinant [7]. We assume that $\gamma_k > 0, \forall k$, and the constraints are feasible.

III. Bound on the Minimum Number of Beamformers for ISAC

This section establishes a bound on the minimum number of beamformers $N_{\min}^{\mathrm{BCRB}}$ needed for communicating to K users while sensing L parameters. We derive such a bound by starting from a full set of N_{T} beamformers, then proving that it can be reduced to a set at most $K + \sqrt{L(L+1)/2}$ beamformers while maintaining the same BFIM and SINR targets. Hereafter, we use the shorthand notation $\mathrm{vSINR}^{(\mathbf{V})} \triangleq \left[\mathrm{SINR}_1^{(\mathbf{V})}, \ldots, \mathrm{SINR}_K^{(\mathbf{V})}\right]^{\mathsf{T}}$ to denote a vector of SINRs.

Theorem 1: Fix a power constraint P and let $\mathcal{A}_P^{(N)}$ denote the set of BFIM-SINR pairs achievable using N beamformers:

$$\mathcal{A}_P^{(N)} \triangleq \left\{ \left(\mathbf{J}^{(\mathbf{V})}, \mathrm{vSINR}^{(\mathbf{V})}\right) \middle| \mathbf{V} \in \mathbb{C}^{N_{\mathrm{T}} \times N}, \mathrm{Tr}(\mathbf{V}\mathbf{V}^{\mathsf{H}}) \leq P \right\}. \tag{9}$$

Then, $\mathcal{A}_P^{(N_{\mathrm{T}})} = \mathcal{A}_P^{(N_{\mathrm{bound}})}$, where

$$N_{\mathrm{bound}} \triangleq \left\lfloor K + \sqrt{\frac{L(L+1)}{2}} \right\rfloor. \tag{10}$$

Thus, the minimum number of beamformers $N_{\min}^{\mathrm{BCRB}}$ for the ISAC problem as expressed in (7) is at most N_{bound}.

Proof: We define an alternative form of the ISAC problem (8) while restricting to using at most N beamformers as follows. For any pair $(\mathbf{J}, \boldsymbol{\gamma}) \in \mathcal{A}_P^{(N)}$, the set of N beamformers that achieves $(\mathbf{J}, \boldsymbol{\gamma})$ with minimum total power is the solution to the following problem:

$$\mathcal{P}^{(N)} : \underset{\mathbf{V} \in \mathbb{C}^{N_{\mathrm{T}} \times N}}{\text{minimize}} \quad \mathrm{Tr}\left(\mathbf{V}\mathbf{V}^{\mathsf{H}}\right) \tag{11a}$$

$$\text{subject to} \quad \mathbf{J}^{(\mathbf{V})} = \mathbf{J}, \tag{11b}$$

$$\mathrm{SINR}_k^{(\mathbf{V})} = \gamma_k, \quad \forall k. \tag{11c}$$

The idea of the proof is to start from $N = N_{\mathrm{T}}$ and some arbitrary $(\mathbf{J}, \boldsymbol{\gamma}) \in \mathcal{A}_P^{(N_{\mathrm{T}})}$ with the optimal solution $\hat{\mathbf{V}}$ for the corresponding $\mathcal{P}^{(N_{\mathrm{T}})}$, and to show that it is always possible to reduce the number of sensing beamformers in $\hat{\mathbf{V}}$ in a sequential fashion until the total number of beamformers is less than or equal to N_{bound}. This is done by applying a special case of an iterative procedure in [9] while maintaining (11b)-(11c) and keeping the same transmit power.

The procedure involves transforming $\hat{\mathbf{V}} \triangleq [\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_K, \hat{\mathbf{V}}_{\mathrm{s}}]$ to some $\mathbf{V}' \triangleq [\mathbf{v}'_1, \ldots, \mathbf{v}'_K, \mathbf{V}'_{\mathrm{s}}]$ with fewer sensing beamformers, by scaling $\hat{\mathbf{v}}_1, \ldots, \hat{\mathbf{v}}_K$ by some appropriate factors and post-multiplying $\hat{\mathbf{V}}_{\mathrm{s}}$ by some "tall" matrix as follows:

$$\mathbf{v}'_k = d_k \hat{\mathbf{v}}_k, \quad d_k \in \mathbb{C}, \quad \forall k, \tag{12}$$

$$\mathbf{V}'_{\mathrm{s}} = \hat{\mathbf{V}}_{\mathrm{s}} \mathbf{U}, \quad \mathbf{U} \in \mathbb{C}^{\hat{m} \times m'}, \quad m' < \hat{m}. \tag{13}$$

Here, $\hat{m}$ and m' denote the number of sensing beamformers before and after multiplication, with $\hat{m} = N_{\mathrm{T}} - K$ initially. The goal is to find $\{d_k\}$ and $\mathbf{U}$ so that the new matrix satisfies

$$\mathbf{J}^{(\mathbf{V}')} = \mathbf{J}, \quad \mathrm{vSINR}_k^{(\mathbf{V}')} = \boldsymbol{\gamma}, \quad \mathrm{Tr}\left(\mathbf{V}'\mathbf{V}'^{\mathsf{H}}\right) \leq P. \tag{14}$$

As a first step, we find $\{d_k\}$ and $\mathbf{U}$ so that the same BFIM and SINRs are maintained. Subsequently, we prove that the $\mathbf{V}'$ so obtained also satisfies the power constraint.

For the transformed $\mathbf{V}'$ to satisfy the BFIM constraint, it must satisfy the following $\frac{L(L+1)}{2}$ quadratic equations due to the symmetric nature of $\mathbf{J}$

$$\mathrm{Tr}\left(\tilde{\mathbf{G}}_{ij}\mathbf{V}'\mathbf{V}'^{\mathsf{H}}\right) = t_{ij}, \quad \forall 1 \leq i \leq j \leq L, \tag{15}$$

with $t_{ij} \triangleq \frac{\sigma^2}{T}\left([\mathbf{J}]_{ij} - [\mathbf{C}]_{ij}\right)$, and further satisfy K equations corresponding to the SINR constraints

$$\frac{|\mathbf{h}_k^{\mathsf{H}}\mathbf{v}'_k|^2}{\gamma_k} - \sum_{n \neq k} |\mathbf{h}_k^{\mathsf{H}}\mathbf{v}'_n|^2 - \mathbf{h}_k^{\mathsf{H}}\mathbf{V}'_{\mathrm{s}}\mathbf{V}'^{\mathsf{H}}_{\mathrm{s}}\mathbf{h}_k = \sigma^2, \quad \forall k. \tag{16}$$

Together, (15)-(16) give rise to the following set of equations in terms of $\{d_k\}$ and $\mathbf{U}$

$$\sum_k |d_k|^2 \hat{\mathbf{v}}_k^{\mathsf{H}}\tilde{\mathbf{G}}_{ij}\hat{\mathbf{v}}_k + \mathrm{Tr}\left(\tilde{\mathbf{G}}_{ij}\hat{\mathbf{V}}_{\mathrm{s}}\mathbf{U}\mathbf{U}^{\mathsf{H}}\hat{\mathbf{V}}_{\mathrm{s}}^{\mathsf{H}}\right) = t_{ij}, \tag{17}$$

$$\frac{|d_k \mathbf{h}_k^{\mathsf{H}}\hat{\mathbf{v}}_k|^2}{\gamma_k} - \sum_{n \neq k} |d_n \mathbf{h}_k^{\mathsf{H}}\hat{\mathbf{v}}_n|^2 - \mathbf{h}_k^{\mathsf{H}}\hat{\mathbf{V}}_{\mathrm{s}}\mathbf{U}\mathbf{U}^{\mathsf{H}}\hat{\mathbf{V}}_{\mathrm{s}}^{\mathsf{H}}\mathbf{h}_k = \sigma^2. \tag{18}$$

Now define a new set of variables $a_k = 1 - |d_k|^2$ and $\mathbf{C} = \mathbf{I} - \mathbf{U}\mathbf{U}^{\mathsf{H}}$. Using the fact that $\hat{\mathbf{V}}$ already satisfies the BFIM and SINR constraints, we can express (17)-(18) in terms of the new variables $\{a_k\}$ and $\mathbf{C}$ as follows:

$$\sum_k a_k \hat{\mathbf{v}}_k^{\mathsf{H}}\tilde{\mathbf{G}}_{ij}\hat{\mathbf{v}}_k + \mathrm{Tr}\left(\tilde{\mathbf{G}}_{ij}\hat{\mathbf{V}}_{\mathrm{s}}\mathbf{C}\hat{\mathbf{V}}_{\mathrm{s}}^{\mathsf{H}}\right) = 0, \tag{19}$$

$$\frac{a_k |\mathbf{h}_k^{\mathsf{H}}\hat{\mathbf{v}}_k|^2}{\gamma_k^*} - \sum_{n \neq k} a_n |\mathbf{h}_k^{\mathsf{H}}\hat{\mathbf{v}}_n|^2 - \mathbf{h}_k^{\mathsf{H}}\hat{\mathbf{V}}_{\mathrm{s}}\mathbf{C}\hat{\mathbf{V}}_{\mathrm{s}}^{\mathsf{H}}\mathbf{h}_k = 0. \tag{20}$$

This is a linear system of equations. In addition, we have the following conditions that arise due to the definitions of $\{a_k\}$ and $\mathbf{C}$:

$$\mathbf{I} - \mathbf{C} \succcurlyeq 0, \quad \mathbf{I} - \mathbf{C} \text{ singular}, \quad 1 - a_k \geq 0, \quad \forall k. \tag{21}$$

It is easy to verify that whenever $\{a_k\}$ and $\mathbf{C}$ satisfy (19), (20) and (21), it is always possible to construct $\mathbf{V}'$ with fewer sensing beamformers and to attain the same BFIM and SINRs. We now show that this is possible whenever

$$\hat{m}^2 > \frac{L(L+1)}{2}. \tag{22}$$

Indeed, the set of equations (19)-(20) comprise a linear homogeneous system with $K + \frac{L(L+1)}{2}$ equations and $K + \hat{m}^2$ real unknowns, because there are K equations from the SINR constraints and $\frac{L(L+1)}{2}$ equations from the BFIM constraint (due to symmetry), and there are K real variables $a_1, \ldots, a_K$ and $\hat{m}^2$ real variables from the Hermitian matrix $\mathbf{C}$. So if (22) holds, the number of unknowns exceeds the number of equations and such system must have a solution $a'_1, \ldots, a'_K, \mathbf{C}'$, which are not all zero. This solution can be scaled as follows

$$\mathbf{C} = \frac{1}{\delta}\mathbf{C}', \quad a_k = \frac{a'_k}{\delta}, \quad \forall k, \tag{23}$$

to additionally satisfy (21). Here, δ is chosen to satisfy

$$|\delta| = \max\{|a'_1|, \ldots, |a'_K|, |\delta'_1|, \ldots, |\delta'_{\hat{m}}|\}, \tag{24}$$

and either $\delta = a'_k$ for some k or $\delta = \delta'_m$ for some m, where $\delta'_1, \ldots, \delta'_m$ are the eigenvalues of $\mathbf{C}'$. Note that $\delta \neq 0$ since $\{a'_k\}$ and $\{\delta'_m\}$ are not all zero.

It is straightforward to verify that $\{a_k\}$ and $\mathbf{C}$ defined in (23) must satisfy $\mathbf{I}-\mathbf{C} \succcurlyeq 0$ and $1-a_k \geq 0, \forall k$. The fact that $\mathbf{I}-\mathbf{C}$ is singular is established by contradiction. Suppose that $\mathbf{I}-\mathbf{C}$ is nonsingular, then by (23), we must have $1 - \frac{\delta_n}{\delta} > 0$ for all $n = \{1, \ldots, \hat{m}\}$, which implies that $\delta = a_i$ for some i. In this case, the corresponding d_i is zero and $\mathbf{v}'_i$ in (12) is the all-zero vector. However, this cannot happen since $\gamma_k > 0, \forall k$.

We now show that $\mathbf{V}'$ obtained by such $\{a_k\}$ and $\mathbf{C}$ has the same power as $\hat{\mathbf{V}}$. This is due to the fact that $\hat{\mathbf{V}}$ is a solution to the problem $\mathcal{P}^{(N_\mathrm{T})}$. First, because $(\mathbf{J}, \boldsymbol{\gamma}) \in \mathcal{A}_P^{(N_\mathrm{T})}$, we must have $\mathrm{Tr}(\hat{\mathbf{V}}\hat{\mathbf{V}}^\mathsf{H}) \leq P$. Furthermore, it turns out that the optimization problem $\mathcal{P}^{(N_\mathrm{T})}$ has strong duality despite being nonconvex, which follows by showing that its SDR is tight using the technique in [3]. Thus, there exist dual variables $\{\nu_{i,j}\}_{1\leq i\leq j\leq L}$ and $\{\mu_k\}_{1\leq k\leq K}$ such that the first-order conditions as written below are satisfied:

$$\hat{\mathbf{v}}_k = \left(\sum_{i\leq j} \nu_{i,j} \tilde{\mathbf{G}}_{i,j} + \frac{\mu_k}{\gamma_k} \mathbf{h}_k \mathbf{h}_k^\mathsf{H} - \sum_{n\neq k} \mu_n \mathbf{h}_n \mathbf{h}_n^\mathsf{H} \right) \hat{\mathbf{v}}_k, \quad \forall k \tag{25}$$

$$\hat{\mathbf{V}}_\mathrm{s} = \left(\sum_{i\leq j} \nu_{i,j} \tilde{\mathbf{G}}_{i,j} - \sum_k \mu_k \mathbf{h}_k \mathbf{h}_k^\mathsf{H} \right) \hat{\mathbf{V}}_\mathrm{s}, \tag{26}$$

and that the primal optimum is equal to the dual optimum

$$\sum_{i\leq j} \nu_{i,j} t_{i,j} + \sigma^2 \sum_k \mu_k = \mathrm{Tr}\left(\hat{\mathbf{V}}\hat{\mathbf{V}}^\mathsf{H}\right). \tag{27}$$

Multiplying both sides of (25) from the left by $|d_k|^2 \hat{\mathbf{v}}_k^\mathsf{H}$ and both sides of (26) from the left by $\hat{\mathbf{V}}_\mathrm{s}^\mathsf{H}\mathbf{U}^\mathsf{H}\mathbf{U}$, and summing the K equations in (25) together with (26), we get

$$\begin{aligned}\mathrm{Tr}\left(\mathbf{V}'^\mathsf{H}\mathbf{V}'\right) &= \sum_{i\leq j} \nu_{i,j} \mathrm{Tr}\left(\mathbf{V}'^\mathsf{H}\tilde{\mathbf{G}}_{i,j}\mathbf{V}'\right) + \sum_k \mu_k \frac{|\mathbf{h}_k^\mathsf{H}\mathbf{v}'_k|^2}{\gamma_k^*} \\ &\quad - \sum_k \mu_k \left(\sum_{n\neq k} |\mathbf{h}_k^\mathsf{H}\mathbf{v}'_n|^2 + \mathbf{h}_k^\mathsf{H}\mathbf{V}'_\mathrm{s}\mathbf{V}_\mathrm{s}'^\mathsf{H}\mathbf{h}_k \right) \\ &= \sum_{i\leq j} \nu_{i,j} t_{i,j} + \sigma^2 \sum_k \mu_k \\ &= \mathrm{Tr}\left(\hat{\mathbf{V}}^\mathsf{H}\hat{\mathbf{V}}\right) \leq P. \end{aligned} \tag{28}$$

where we make use of (15) and (16). Thus, $\mathbf{V}'$ has power at most P. To summarize, provided that the number of sensing beamformers satisfies (22), it is always possible to reduce the number of sensing beamformers while satisfying (14).

Now, denote the new total number of beamformers by $N' = K + m'$ and consider the problem $\mathcal{P}^{(N')}$. We now make two observations. First, we claim that $\mathbf{V}'$ is an optimal solution of $\mathcal{P}^{(N')}$. This is because the optimal value of $\mathcal{P}^{(N')}$ is at least that of $\mathcal{P}^{(N_\mathrm{T})}$, but $\mathbf{V}'$ has the same power as $\hat{\mathbf{V}}$, so $\mathbf{V}'$ must achieve the minimum power of $\mathcal{P}^{(N')}$.

Second, we claim that strong duality must also hold for $\mathcal{P}^{(N')}$. This is because $\mathcal{P}^{(N')}$ and $\mathcal{P}^{(N_\mathrm{T})}$ achieve the same primal optimal value as shown earlier, and further, they also have the same SDR. The SDR of this type of nonconvex optimization problem gives the optimal value of its dual problem. Combined with the strong duality of $\mathcal{P}^{(N_\mathrm{T})}$, this shows that $\mathcal{P}^{(N')}$ also has strong duality, despite being nonconvex.

With this in mind, if the new number of sensing beamformers matrix m' also satisfies (22), we can set $\mathbf{V}'$ as the new $\hat{\mathbf{V}}$ and m' as the new $\hat{m}$, and repeat the process. Due to the two key observations mentioned above, it can be verified that every step of this sensing beamformer reduction process can be carried out as before. This procedure continues until the number of sensing beamformers is less than or equal to $\lfloor\sqrt{L(L+1)/2}\rfloor$, or equivalently, the total number of beamformers is less than or equal to N_bound. Since $(\mathbf{J}, \boldsymbol{\gamma})$ can be any point in $\mathcal{A}^{(N_\mathrm{T})}$, this shows that $\mathcal{A}_P^{(N_\mathrm{T})} = \mathcal{A}_P^{(N_\text{bound})}$.

Finally, since the BFIM and SINRs corresponding to the solution of the ISAC problem (8) always lie in $\mathcal{A}_P^{(N_\mathrm{T})}$, such solution can always be achieved using at most N_bound beamformers. This shows that N_min^BCRB is at most N_bound. ■

We remark that the proof of Theorem 1 does not follow directly from the results in [9]. Moreover, the proof of Theorem 1 reveals that the bound (10) is not tied to the specific ISAC optimization used to define N_min^BCRB. This is because a different optimization must also yield some $(\mathbf{J}, \boldsymbol{\gamma})$ in the achievable set so that the same bound remains applicable.

IV. Bound on the Minimum Number of Beamformers for Sensing LoS Targets

In many practical sensing applications, such as that of estimating the complex path losses and the AoAs of LoS targets, due to the structural properties of the BFIM, the bound on the minimum number of beamformers can be further tightened. The key to obtaining a tighter bound is to consider the *effective* number of quadratic terms in the BFIM. This section formalizes this concept by introducing the notion of M-quadratic functions.

Definition 1: A real function of matrix $h_M(\mathbf{V}) : \mathbb{C}^{N_\mathrm{T}\times N} \to \mathbb{R}$ is said to be M-quadratic, if it has the following form:

$$h_M(\mathbf{V}) = g\left(\mathrm{Tr}\left(\mathbf{Q}_1\mathbf{V}\mathbf{V}^\mathsf{H}\right), \ldots, \mathrm{Tr}\left(\mathbf{Q}_M\mathbf{V}\mathbf{V}^\mathsf{H}\right)\right), \tag{29}$$

where $\mathbf{Q}_1, \ldots, \mathbf{Q}_M$ are distinct nonzero Hermitian matrices and $g(\cdot)$ is a function from $\mathbb{R}^M$ to $\mathbb{R}$.

In the ISAC context, a sensing performance metric is M-quadratic if it depends on the design variables (i.e., the beamforming matrix $\mathbf{V}$) only through M distinct quadratic functions of $\mathbf{V}$. The BCRB-based sensing metrics are examples of an M-quadratic function. Consider the optimization objective for the ISAC problem (8). Assuming that the matrices $\{\tilde{\mathbf{G}}_{ij}\}$ in (6) are all distinct and nonzero, based on (5)-(6), the BFIM can be expressed as

$$\mathbf{J}^{(\mathbf{V})} = r\left(\mathrm{Tr}\left(\tilde{\mathbf{G}}_{11}\mathbf{V}\mathbf{V}^\mathsf{H}\right), \ldots, \mathrm{Tr}\left(\tilde{\mathbf{G}}_{LL}\mathbf{V}\mathbf{V}^\mathsf{H}\right)\right) \tag{30}$$

where $r(\cdot)$ is a function that arranges its arguments in an $L \times L$ symmetric matrix with a constant matrix $\mathbf{C}$ added to it, i.e., for $\mathbf{w} = [w_{11}, \ldots, w_{1L}, w_{22}, \ldots w_{2L}, \ldots w_{LL}]^\mathsf{T} \in \mathbb{R}^{L(L+1)/2}$,

$$r(\mathbf{w}) = \mathbf{C} + \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1L} \\ w_{12} & w_{22} & \cdots & w_{2L} \\ \vdots & \vdots & \ddots & \vdots \\ w_{1L} & w_{2L} & \cdots & w_{LL} \end{bmatrix}. \quad (31)$$

Then, the objective of (8) can be seen as an M-quadratic function with $M = \frac{L(L+1)}{2}$.

In many practical ISAC scenarios, some of $\{\tilde{\mathbf{G}}_{ij}\}$ are zero or are repeated in different entries of the BFIM. In these cases, the objective of (8) can still be viewed as an M-quadratic function, but with M being strictly less than $\frac{L(L+1)}{2}$. This leads to a tighter bound on the minimum number of beamformers needed for ISAC. In the rest of this section, we first derive a new bound on the minimum number of beamformers needed when the sensing metric is an M-quadratic function, then give an example of the tighter bound.

Theorem 2: Fix a power constraint P and assume that the sensing metric $h_M(\cdot)$ is an M-quadratic function. Let $\tilde{\mathcal{A}}_P^{(N)}$ denote the set of sensing-metric-value and SINRs, achievable using N beamformers, i.e.,

$$\tilde{\mathcal{A}}_P^{(N)} \triangleq \left\{ \left(h_M(\mathbf{V}), \text{vSINR}^{(\mathbf{V})}\right) \middle| \mathbf{V} \in \mathbb{C}^{N_\text{T} \times N}, \text{Tr}(\mathbf{V}\mathbf{V}^\mathsf{H}) \le P \right\}. \quad (32)$$

Then, $\tilde{\mathcal{A}}_P^{(N_\text{T})} = \tilde{\mathcal{A}}_P^{(N_\text{bound}^{(M)})}$, where

$$N_\text{bound}^{(M)} \triangleq \lfloor K + \sqrt{M} \rfloor. \quad (33)$$

Proof: The proof follows the same line of argument as Theorem 1, except when counting the number of linear equations, there are now $K + M$ equations instead of $K + \frac{L(L+1)}{2}$ equations. The details are omitted due to space limitation. ■

Theorem 2 reduces to Theorem 1 when $M = \frac{L(L+1)}{2}$. The following example shows how M can be significantly less in specific applications. Consider the scenario of sensing N LoS targets characterized by a round-trip channel

$$\mathbf{G}^{(\boldsymbol{\eta})} = \sum_{i=1}^{N} \alpha_i \mathbf{A}(\theta_i), \quad (34)$$

where α_i's are the complex path losses and θ_i's are the AoAs. There are a total of $L = 3N$ parameters:

$$\boldsymbol{\eta} = \left[\Re\{\alpha_1\}, \Im\{\alpha_1\}, \theta_1, \ldots, \Re\{\alpha_N\}, \Im\{\alpha_N\}, \theta_N\right]^\mathsf{T}. \quad (35)$$

But, instead of bounding $N_\text{min}^\text{BCRB} \le K + \left\lfloor \sqrt{\frac{3N(3N+1)}{2}} \right\rfloor$, we can view the sensing metric as an M-quadratic function, because the BFIM can be written as

$$\mathbf{J}^{(\mathbf{V})} = \mathbf{C} + \begin{bmatrix} \mathbf{T}_{11}^{(\mathbf{V})} & \cdots & \mathbf{T}_{1N}^{(\mathbf{V})} \\ \vdots & \ddots & \vdots \\ \mathbf{T}_{N1}^{(\mathbf{V})} & \cdots & \mathbf{T}_{NN}^{(\mathbf{V})} \end{bmatrix} \quad (36)$$

where $\mathbf{T}_{ii}^{(\mathbf{V})}$ and $\mathbf{T}_{ij}^{(\mathbf{V})}, i \neq j$ are 3×3 matrices whose elements correspond to the *intra-traget* and *inter-target* parameters, respectively. In particular, the matrix $\mathbf{T}_{ii}^{(\mathbf{V})}$ is given by [2]

$$\mathbf{T}_{ii}^{(\mathbf{V})} = \begin{bmatrix} \text{Tr}(\mathbf{Q}_{ii,1}\mathbf{V}\mathbf{V}^\mathsf{H}) & 0 & \text{Tr}(\mathbf{Q}_{ii,3}\mathbf{V}\mathbf{V}^\mathsf{H}) \\ 0 & \text{Tr}(\mathbf{Q}_{ii,1}\mathbf{V}\mathbf{V}^\mathsf{H}) & \text{Tr}(\mathbf{Q}_{ii,4}\mathbf{V}\mathbf{V}^\mathsf{H}) \\ \text{Tr}(\mathbf{Q}_{ii,3}\mathbf{V}\mathbf{V}^\mathsf{H}) & \text{Tr}(\mathbf{Q}_{ii,4}\mathbf{V}\mathbf{V}^\mathsf{H}) & \text{Tr}(\mathbf{Q}_{ii,2}\mathbf{V}\mathbf{V}^\mathsf{H}) \end{bmatrix}$$

where $\mathbf{Q}_{ii,1}, \ldots, \mathbf{Q}_{ii,4}$ are Hermitian matrices. Note that instead of having 6 distinct terms (due to its symmetry), the zero and the repeated element $\text{Tr}(\mathbf{Q}_{ii,1}\mathbf{V}\mathbf{V}^\mathsf{H})$ reduce the number of distinct terms in $\mathbf{T}_{ii}^{(\mathbf{V})}$ to 4. Likewise, it can be shown that each of the matrices $\mathbf{T}_{ij}^{(\mathbf{V})}$ has 7 distinct terms, so that $M = 4N + \frac{7}{2}N(N-1)$. Applying Theorem 2, we obtain the improved bound

$$N_\text{min}^\text{BCRB} \le K + \left\lfloor \sqrt{3.5N^2 + 0.5N} \right\rfloor. \quad (37)$$

If $K = 0$, we obtain an improvement on the $2N$ bound in [7] derived under the classical CRB. Interestingly, this shows that the minimum number of beamformers grows asymptotically at most as $1.871N$ for sensing $3N$ parameters.

A key advantage of the more general bound in Theorem 2 is that it is not only limited to BCRB as the sensing metric. There are many other examples of M-quadratic functions that are relevant to estimation and detection tasks, for instance, the radar-SNR [6], metric for beam pattern matching [3], etc.

V. Conclusion

This paper introduces nontrivial bounds on the minimum number of beamformers required for integrated sensing and communications. The bounds of this paper are applicable to any number of users and sensing parameters and can be further applied to a large family of sensing metrics that have quadratic dependence on the beamformers.

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