Modulo Quantization Coding for Gaussian Primitive Relay Channel with Perfectly Correlated Noises

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***Abstract*—The Gaussian primitive relay channel (i.e., a relay channel with a noiseless relay-to-receiver link) with perfectly correlated noises exhibits an intriguing phenomenon: it is possible to transmit at a strictly positive rate even at infinitesimal transmit power. This paper introduces a simple modulo-quantization-based scheme that achieves zero-error communication for the considered channel, even at arbitrarily small (but non-zero) power. This scheme operates at a rate equal to the relay-to-receiver link capacity. Furthermore, the proposed scheme can be superposed onto a Gaussian code defined on an integer lattice to achieve the capacity of the Gaussian primitive relay channel with perfectly correlated noise at any finite input power.**

I. Introduction

The capacity of an additive white Gaussian noise channel under power constraint P, as depicted in Fig. 1(a), is $\psi(P) \triangleq \frac{1}{2}\log(1+P)$. As $P \to 0$, the capacity $\psi(P) \to 0$, i.e., the achievable rate of reliable transmission goes down to zero. Intriguingly, if the Gaussian channel is equipped with a relay that can observe the noise Z perfectly and has a noiseless relay-to-receiver link of finite positive capacity R_0 (see Fig. 1(b)), it can be shown that even at arbitrarily small (but non-zero) power, the transmitter would be able to communicate to the receiver reliably at a positive rate [1, Remarks 2 & 3], [2]. In fact, as P approaches 0, the capacity of this channel tends to the relay link capacity R_0.

This intriguing phenomenon arises because of the perfect correlation between the noises at the relay and the receiver. A generalization of the model in which this phenomenon can be observed is depicted in Fig. 1(c). Here, the relay observes Y_0. The receiver observes Y. Each is obtained by adding the same Gaussian noise Z to some scaled version of the power-constrained input X, with scaling factors α and β, respectively. A noiseless relay link of capacity R_0 connects the relay to the receiver. Assuming that the channel gains of the relay and the receiver are different, the capacity of this relay channel is $\psi(\beta^2 P) + R_0$, i.e., the sum of the capacities of the direct (transmitter-to-relay) channel and of the noiseless link [3]. As the transmit power goes to zero, the capacity tends to R_0. In other words, the phenomenon discussed in the last paragraph is again observed.

Perfectly (or, at least, near-perfectly) correlated noises can occur in practice. Consider a setting where multiple receivers are affected by a common and significant interference source. When the interference massively overpowers the receiver noise, the correlation of the effective noises approaches one.

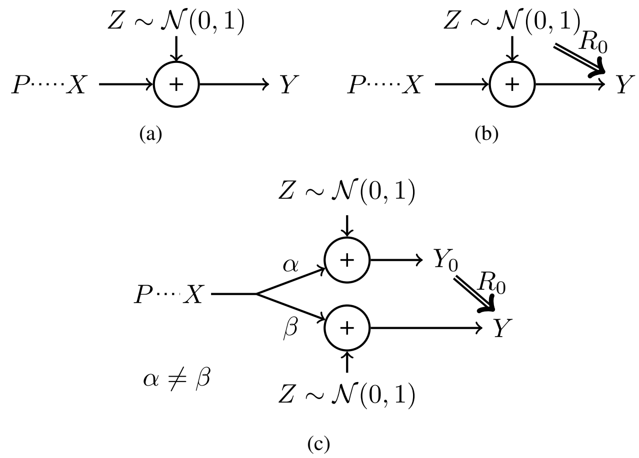

Fig. 1. (a) Gaussian channel with power constraint P; (b) Gaussian channel with a primitive relay link of finite capacity R_0. (c) A general model of Gaussian primitive relay channel with perfectly correlated noises.

The goal of this paper is to design a coding scheme that can exploit the perfect noise correlation and achieve the capacity of the channel in Fig. 1(c). This is done in two steps. First we describe a particularly simple zero-error scalar coding scheme called *modulo quantization coding*. This scheme achieves the rate R_0 at any positive transmit power (which is the asymptotic capacity in the low-power limit). The main idea is to forward the modulo quantization of the relay's observation to the receiver through the noiseless link. The receiver is able to recover the transmitted message perfectly by performing modular arithmetic operations on the relay's message and the modulo quantization of its own observation.

The previous coding scheme can be extended by superposing the modulo quantization scheme on a Gaussian channel code supported on a scaled integer lattice. The rate of the resulting *superposed modulo quantization code* equals R_0 plus the rate of the Gaussian channel code. This is because incorporating the modulo quantization scheme does not introduce any error, and it is possible to construct a sequence of Gaussian channel codes supported on a lattice with rates approaching the direct channel capacity and probability of error approaching zero. This superposed modulo quantization coding strategy is in fact capacity achieving for the Gaussian primitive relay channel in Fig. 1(c).

Modulo operations play important roles in a variety of practical communication strategies. A well-known example is Tomlinson-Harashima precoding (THP) [4], [5], which is a practical scalar implementation of dirty paper coding [6]. However, the purpose of the modulo operation in THP differs from that proposed herein. In THP, the main purpose of the modulo operator is to reduce the transmit power. In contrast, the main purpose of the modulo quantizer in this paper is to reduce the rate of the relay transmission.

II. Problem Formulation

Consider the channel model depicted in Fig. 1(c). The transmitter sends signal X subject to power constraint $\mathbb{E}[X^2] \leq P$. The relay observation Y_0 and the receiver observation Y are obtained by passing scaled versions of X through additive white Gaussian channels with identical noises Z. This model is a special case of *primitive relay channel*, i.e., the relay can communicate to the receiver via a separate noiseless link with capacity R_0. Specifically, the channel model is defined by

$$\begin{cases} Y_0 = \alpha X + Z \\ Y = \beta X + Z \end{cases}, \tag{1}$$

where $Z \sim \mathcal{N}(0,1)$ is a standard normal random variable, and $\alpha, \beta \in \mathbb{R}$ are distinct real-valued channel gains, i.e., $\alpha \neq \beta$. We require the channel gains to be distinct, because otherwise the relay would have the exact same observation as the receiver, and relaying would be useless.

For this channel model, an (n,R)-code consists of an encoding function $f^{(n)} : [2^{nR}] \to \mathbb{R}^n$, a relay function $\rho^{(n)} : \mathbb{R}^n \to [2^{nR_0}]$, and a decoding function $g^{(n)} : \mathbb{R}^n \times [2^{nR_0}] \to [2^{nR}]$, where $[k]$ stands for the integer set $\{0, 1, \cdots, k-1\}$. Let $X^n = f^{(n)}(M)$ be the input, where M is the message index uniformly distributed over $[2^{nR}]$. Let Y_0^n and Y^n be the respective observations of the relay and the receiver. The probability of decoding error is defined by $P_e^{(n)} = \Pr\{M \neq g^{(n)}(Y^n, \rho^{(n)}(Y_0^n))\}$, The capacity $C(R_0)$ is the supremum of rates such that the error probability $P_e^{(n)}$ can be made to approach zero as $n \to \infty$.

The capacity of this channel is known. Observe that the relay observation Y_0 is a deterministic function of the transmitted signal X and the receiver observation Y, i.e., there exists a function $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$Y_0 = \varphi(X, Y). \tag{2}$$

The capacity of primitive relay channels for which the deterministic relation (2) holds is given in [3] as follows:

$$C = \sup_{p(x)} \min\{I(X;Y) + R_0, I(X;Y,Y_1)\}. \tag{3}$$

By specializing (3) to the channel model in Fig. 1(c), we have the following theorem.

Theorem 1. *The capacity of the deterministic Gaussian primitive relay channel depicted in Fig. 1(c) is*

$$C(R_0) = C(0) + R_0 = \psi(\beta^2 P) + R_0. \tag{4}$$

In particular, as $P \to 0$, $C(R_0) \to R_0$.

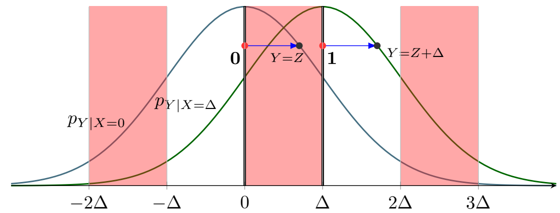

Fig. 2. Illustration of the coding scheme for the model in Fig. 1(b).

The capacity of deterministic primitive relay channels satisfying (3) can be achieved by compress-and-forward (CF), introduced by Cover and El Gamal in [7]. In the CF scheme, Wyner-Ziv coding [8] is used to compress the relay observation, while accounting for the receiver observation (as side information for source coding), and the compressed observation is then sent to the receiver. The receiver recovers the transmitted message based on its own observation and the compressed version of the relay's observation. However, the vector nature of the quantization involved makes the implementation of CF very complex.

An alternative scheme called hash-and-forward (HF) is proposed in [9] to achieve the capacity for deterministic primitive relay channels. However, this scheme involves performing random hashing on all possible relay observations, making it impossible to directly implement for channels defined over continuous alphabets, wherein the space of relay observations is uncountably large.

The main question this paper aims to answer is the following. If we restrict attention to the Gaussian deterministic primitive relay channel, i.e., the channel model in Fig. 1(c), is it possible to design a low-complexity coding scheme to achieve the channel capacity?

III. Modulo Quantization Coding Schemes

A. Motivating Example

To help understand the proposed schemes, consider first the channel model depicted in Fig. 1(b), which is a special case of Fig. 1(c) with $\alpha = 0$ and $\beta = 1$. For pedagogical simplicity, suppose the relay can transmit one bit to the receiver per channel use, i.e., $R_0 = 1$. We now show how to leverage the relay link to convey 1 bit of information from the transmitter to the receiver in each channel use in a zero-error fashion, regardless of the (non-zero) transmit power.

The proposed approach is as follows. First, fix any $\Delta > 0$. Partition the real line into contiguous intervals of length Δ and color them alternately with white and gray as shown in Fig. 2. In each channel use, the transmitter sends 1 bit of information by setting $X = 0$ or $X = \Delta$. Observe that if $X = 0$ then $Y = Z$ while if $X = \Delta$, $Y = Z + \Delta$. Now, the relay observes $Y_0 = Z$. If the relay and the receiver compare the colors of the intervals in which their observations are located, they get matching colors if $X = 0$ (since $Y = Y_0 = Z$); otherwise they get different colors if $X = \Delta$ (since $Y_0 = Z$, $Y = Z + \Delta$ and the interval-width is Δ). This holds regardless of the realization of the noise Z. Hence, if the relay sends to the

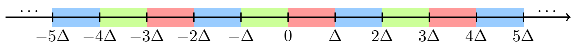

Fig. 3. Illustration of the modulo quantizer $\lceil\cdot\rfloor_{\Delta,3}$. Each color corresponds to one modulo quantization output.

receiver the color of the interval in which $Y_0 = Z$ is situated, upon receiving Y, the receiver would be able to decode X by checking whether Y and Y_0 belong to same color intervals.

The scheme is error-free regardless of the value of Δ, as long as Δ is strictly positive. Hence Δ can be made arbitrarily small. The average transmit power of this scheme is $P = \Delta^2/2$. Thus, infinitesimally small amount of transmit power already suffices to attain error-free communication of 1 bit per channel use for this channel.

B. Modulo Quantization Coding

The idea in the previous subsection can be generalized to construct a coding scheme for the model in Fig. 1(c). In this section, we propose a modulo quantization coding scheme to achieve rate R_0 with arbitrary non-zero power.

We first introduce some notation. Fix positive real number $\Delta > 0$ and positive integer $k \in \mathbb{N}$. The (Δ, k)-*modulo quantizer* is the function $\lceil\cdot\rfloor_{\Delta,k} : \mathbb{R} \to [k]$ defined as

$$\lceil x\rfloor_{\Delta,k} := \left\lfloor \frac{x}{\Delta} \right\rfloor \bmod k. \tag{5}$$

The floor function $\lfloor\cdot\rfloor : \mathbb{R} \to \mathbb{Z}$ returns the largest integer less than or equal to its argument. The modulo-k operation $(\cdot) \bmod k : \mathbb{Z} \to [k]$ returns the remainder of the argument divided by k.

The modulo quantization operation can be interpreted as follows: $\lfloor \frac{x}{\Delta} \rfloor$ is the output of an infinite-support scalar quantizer with step-size Δ with input x, and $\lceil x\rfloor_{\Delta,k}$ can be seen as the mod-k "wrapped" version of the quantized output. Alternatively, $\lceil x\rfloor_{\Delta,k}$ can be viewed as an instance of one-dimensional deterministic binning, where points are put into bins according to the colors of the intervals they belong to, as illustrated in Fig. 3. Using this notation, the relay's strategy in Section III-A can be expressed as sending the modulo quantization $\lceil Z\rfloor_{\Delta,2}$ to the receiver.

A few properties of the floor function and the modulo operation are listed below. For all $x \in \mathbb{R}$, $a, b \in \mathbb{Z}$, and $k \in \mathbb{N}$, the following relations hold:

$$\lfloor x + a\rfloor = \lfloor x\rfloor + a, \tag{6}$$
$$(a \bmod k) \bmod k = a \bmod k, \tag{7}$$
$$(a + b) \bmod k = [(a \bmod k) + b] \bmod k. \tag{8}$$

The following lemma is a direct consequence of the above.

Lemma 1. *For any $a \in \mathbb{Z}$, $\Delta > 0$ and $k \in \mathbb{N}$,*

$$\lceil x + a\Delta\rfloor_{\Delta,k} = (\lceil x\rfloor_{\Delta,k} + a) \bmod k. \tag{9}$$

In particular,

$$\lceil x + ak\Delta\rfloor_{\Delta,k} = \lceil x\rfloor_{\Delta,k}. \tag{10}$$

The modulo quantization coding scheme is specified as follows. For now, assume $R_0 = \log k$ for some positive integer $k \in \mathbb{N}$. The quantity k is the number of messages that the relay can communicate to the receiver per channel use. We also let k be the size of the transmitter's message set, so the overall rate of this scheme is also R_0.

Let $\Delta > 0$ be a positive constant to be decided later. To transmit a message $M \in [k]$, the transmitter sets $X = M\Delta$. The relay observes $Y_0 = \alpha X + Z$ and sends $U_0 = \lceil Y_0\rfloor_{|\beta-\alpha|\Delta,k}$, the $(|\beta - \alpha|\Delta, k)$-modulo quantization of its observation, to the receiver. The receiver observes $Y = \beta X + Z$ and computes $U = \lceil Y\rfloor_{|\beta-\alpha|\Delta,k}$. It then declares the transmitted message to be $\hat{M} = (U - U_0) \bmod k$ if $\beta > \alpha$, or $\hat{M} = (U_0 - U) \bmod k$ if $\beta < \alpha$.

The above scalar scheme is error-free, i.e., $\hat{M} = M$ always holds. First consider the case where $\beta > \alpha$, in which

$$\begin{aligned} Y &= \beta X + Z && (11)\\ &= (\alpha X + Z) + (\beta - \alpha)X && (12)\\ &= Y_0 + (\beta - \alpha)M\Delta. && (13) \end{aligned}$$

By Lemma 1,

$$\begin{aligned} U &= \lceil Y_0 + M \cdot (\beta - \alpha)\Delta\rfloor_{(\beta-\alpha)\Delta,k} && (14)\\ &= (\lceil Y_0\rfloor_{(\beta-\alpha)\Delta,k} + M) \bmod k && (15)\\ &= (U_0 + M) \bmod k. && (16) \end{aligned}$$

The decoded message is thus exactly the transmitted message:

$$\begin{aligned} \hat{M} &= (U - U_0) \bmod k && (17)\\ &= ((M + U_0) \bmod k - U_0) \bmod k && (18)\\ &= (M + U_0 - U_0) \bmod k && (19)\\ &= M, && (20) \end{aligned}$$

where (19) is due to (8), and (20) is due to the fact that $M \in [k]$ implies $M \bmod k = M$. For the case where $\alpha > \beta$, a similar calculation gives $\hat{M} = (U_0 - U) \bmod k = M$.

Let P be an arbitrarily small positive power. Set $\Delta = \sqrt{3P/k^2}$. Since the k messages are equiprobable, the average transmission power is upper bounded as:

$$\frac{1}{k}\sum_{m=0}^{k-1}(m\Delta)^2 = \frac{1}{6}(k-1)(2k-1)\Delta^2 \le \frac{k^2\Delta^2}{3} = P. \tag{21}$$

The power constraint is therefore satisfied.

When $k = 2^{R_0}$ is not an integer, the rate R_0 can be achieved using time-sharing. Find positive integers $k_1, k_2 \in \mathbb{N}$ and $n_1, n_2 \in \mathbb{N}$ such that

$$R' = \frac{n_1}{n_1 + n_2}\log k_1 + \frac{n_2}{n_1 + n_2}\log k_2 \tag{22}$$

is arbitrarily close to R_0 from below. For $i = 1, 2$, use the above scalar scheme with k_i messages and $\Delta_i = \sqrt{3P/k_i^2}$ for n_i channel uses. The average power constraint is satisfied. The time-shared relay rate is $R' \le R_0$, and the time-shared overall achieved rate is R', which can approach R_0.

We summarize the above results in the following theorem.

Theorem 2. *For the deterministic Gaussian primitive relay channel depicted in Fig. 1(c), the rate R_0 is achievable using modulo quantization coding at any average transmit power $P > 0$.*

C. Superposed modulo quantization coding

In the last subsection, a simple coding scheme that achieves capacity for the deterministic Gaussian primitive relay channel in the low-power limit is developed. In order to achieve capacity at any power, we propose the following extension called *superposed modulo quantization coding*.

The main idea of the extended scheme is to encode two messages simultaneously into one signal. The transmitted message is separated into two parts, which are encoded independently via separate codebooks into auxiliary sequences. The transmitted codeword corresponding to the message is obtained by superposing the two auxiliary sequences. In particular, one part of the message is encoded via a Gaussian codebook supported on a scaled integer lattice, and the other part is encoded via the previously described modulo quantization codebook, which resides on a finer lattice. Note that even though two lattices are used here, the proposed structure is quite different from the nested lattice code [10], [11]. In nested lattice coding, only the finer lattice is used for channel coding, and the coarser lattice is used for source coding; however in our proposed scheme, both lattices are used for channel coding.

The relay computes the (symbol-wise) modulo quantization of its received signal and forwards it to the receiver. The receiver is able to decode the message encoded with the modulo quantization codebook perfectly by comparing the relay's transmission to the modulo quantization of its own observed signal. It can then subtract off the effect of the modulo quantization codebook and proceed to decode the message encoded with the Gaussian codebook. The coding process is reminiscent of the superposition coding technique and the successive cancellation decoding method for the broadcast channel [12].

We now specify the superposed modulo quantization coding scheme in detail. Let $\tilde{\mathcal{C}}$ be an $(n, \tilde{R}, \tilde{P})$-Gaussian code for the direct channel, i.e., the channel with input X and output $Y = \beta X + Z$. The parameters $n, \tilde{R}, \tilde{P}$ respectively stand for the blocklength, rate, and maximum codeword power of the code. Specifically, let the codewords of $\tilde{\mathcal{C}}$ be $\{\tilde{X}^n(1), \cdots, \tilde{X}^n(2^{n\tilde{R}})\}$ and the decoding function be $\tilde{g} : \mathbb{R}^n \to [2^{n\tilde{R}}]$. The squared norm of each codeword is upper bounded by $\tilde{P}$, i.e.,

$$\frac{1}{n}\|\tilde{X}^n(\tilde{M})\|^2 \le \tilde{P},\ \forall\, \tilde{M} \in [2^{n\tilde{R}}]. \tag{23}$$

We require that the code $\tilde{\mathcal{C}}$ to be supported on a scaled integer lattice, i.e., there exists $\tilde{\Delta} > 0$ such that

$$\tilde{X}^n(\tilde{M}) \in (\tilde{\Delta}\mathbb{Z})^n,\ \forall\, \tilde{M} \in [2^{n\tilde{R}}]. \tag{24}$$

As a remark, the scaling factor $\tilde{\Delta}$ can be chosen to be arbitrarily small, since

$$\tilde{\mathcal{C}} \subset (\tilde{\Delta}\mathbb{Z})^n \quad \Rightarrow \quad \tilde{\mathcal{C}} \subset (\tfrac{\tilde{\Delta}}{a}\mathbb{Z})^n,\ \forall\, a \in \mathbb{N}. \tag{25}$$

The average probability of error for the code $\tilde{\mathcal{C}}$ on the direct channel $Y = \beta X + Z$ is denoted by $P_e(\tilde{\mathcal{C}})$:

$$P_e(\tilde{\mathcal{C}}) := 2^{-n\tilde{R}} \sum_{i=1}^{2^{n\tilde{R}}} \Pr\{i \ne \tilde{g}(\beta\tilde{X}^n(i) + Z^n)\}. \tag{26}$$

Given an $(n, \tilde{R}, \tilde{P})$ lattice-support Gaussian code $\tilde{\mathcal{C}}$, we show in the following how to construct an $(n, \tilde{R}+R_0, P)$-code for the relay channel depicted in Fig. 1(c) for any $P > \tilde{P}$ with error probability $P_e = P_e(\tilde{\mathcal{C}})$.

Consider first the case where $R_0 = \log k$ for some $k \in \mathbb{N}$. Identify the message set $[2^{n(\tilde{R}+R_0)}]$ with the Cartesian product $[2^{n\tilde{R}}] \times [k]^n$, i.e., consider messages of the form $(\tilde{m}, m^n)$, where $\tilde{m} \in [2^{n\tilde{R}}]$ and $m^n = (m_1, \cdots, m_n) \in [k]^n$.

As remarked above, let $\tilde{\Delta} > 0$ be such that $\tilde{\mathcal{C}} \subset (\tilde{\Delta}\mathbb{Z})^n$. Set $\tilde{P} \le P - \tilde{\Delta}^2$. Let $\Delta := \tilde{\Delta}/k$. The proposed encoding function $f^{(n)} : [2^{n\tilde{R}}] \times [k]^n \to \mathbb{R}^n$ is

$$f^{(n)}(\tilde{m}, m^n) = X^n(\tilde{m}, m^n) := \tilde{X}^n(\tilde{m}) + \Delta \cdot m^n. \tag{27}$$

The above choice of $\tilde{\Delta}$ ensures that the power constraint is satisfied for the relay channel, i.e., $\forall\,(\tilde{m}, m^n) \in [2^{n\tilde{R}}] \times [k]^n$,

$$\begin{aligned}
\tfrac{1}{n}\|X^n(\tilde{m}, m^n)\|^2 &\le \tfrac{1}{n}(\|\tilde{X}^n(\tilde{m})\|^2 + \Delta^2\|m^n\|^2) && (28)\\
&\le \tilde{P} + \tfrac{1}{n} \cdot nk^2\Delta^2 && (29)\\
&\le (P - \tilde{\Delta}^2) + \tilde{\Delta}^2 && (30)\\
&= P. && (31)
\end{aligned}$$

Given message $(\tilde{M}, M^n)$, the relay observes $Y_0^n = \alpha X^n(\tilde{M}, M^n) + Z^n$. The relay forwards the symbol-wise $(|\beta-\alpha|\Delta, k)$-modulo quantization U_0^n to the receiver, where $U_{0,i} = \lceil Y_{0,i} \rfloor_{|\beta-\alpha|\Delta,k}, \forall\, 1 \le i \le n$. The receiver receives U_0^n from the relay and observes $Y^n = \beta X^n(\tilde{M}, M^n) + Z^n$. It first aims to recover M^n. To do this, it computes the symbol-wise $(|\beta-\alpha|\Delta, k)$-modulo quantization U^n of its received signal Y^n, i.e., $U_i = \lceil Y_i \rfloor_{|\beta-\alpha|\Delta,k}, \forall\, 1 \le i \le n$. Then it declares

$$\hat{M}_i = (\operatorname{sgn}(\beta-\alpha) \cdot (U - U_0)) \bmod k. \tag{32}$$

The receiver is able to decode the M^n part of the message perfectly, i.e., $\hat{M}_i = M_i$ holds for all $1 \le i \le n$. Consider first the case where $\beta > \alpha$. For any i,

$$Y_i = Y_{0,i} + (\beta-\alpha) \cdot (\tilde{X}_i(\tilde{M}) + \Delta \cdot M_i) \tag{33}$$

is obtained in a similar fashion to (11)-(13). The lattice support condition (24) ensures $\tilde{X}_i(\tilde{M}) \in \tilde{\Delta}\mathbb{Z}$. Since $\tilde{\Delta} = k\Delta$, there exists some $a \in \mathbb{Z}$ such that $(\beta-\alpha) \cdot \tilde{X}_i(\tilde{M}) = ak(\beta-\alpha)\Delta$. By (10) in Lemma 1,

$$\begin{aligned}
U_i &= \lceil Y_{0,i} + (\beta-\alpha) \cdot (\tilde{X}_i(\tilde{M}) + \Delta \cdot M_i) \rfloor_{(\beta-\alpha)\Delta,k} && (34)\\
&= \lceil Y_{0,i} + (\beta-\alpha)\Delta \cdot M_i) \rfloor_{(\beta-\alpha)\Delta,k}, && (35)
\end{aligned}$$

which further simplifies to the following (cf. (14)-(16)):

$$U_i = (U_{0,i} + M_i) \bmod k. \tag{36}$$

Using the same argument in (17)-(20),we conclude that $\hat{M}_i = M_i$. When $\alpha > \beta$, a similar calculation shows $\hat{M}_i = M_i$.

Upon recovering M^n, the receiver can subtract $\beta\Delta \cdot M^n$ from Y^n and get

$$\tilde{Y}^n = Y^n - \beta\Delta \cdot M^n = \beta\tilde{X}^n(\tilde{M}) + Z^n. \tag{37}$$

Observe that $\tilde{Y}^n$ is exactly the output of the direct channel when $\tilde{X}^n(\tilde{M})$ is sent. Averaging over the choice of $\tilde{M}$, the receiver can decode $\tilde{M}$ correctly with error probability $P_e(\tilde{\mathcal{C}})$ by (26). Since the recovery of M^n is perfect, the overall average error probability for this superposed modulo quantization coding scheme is $P_e = P_e(\tilde{\mathcal{C}})$.

To see that the capacity (4) is indeed achievable, it suffices to show the existence of a sequence of capacity-achieving codes supported on lattice for the direct channel. Using a random coding argument, the problem can be reduced to finding a sequence of distributions $\{\mu^{(j)}\}_{j\in\mathbb{N}}$ with support on some one-dimensional lattice and second moment strictly less than P, such that the following mutual information converges to the Gaussian channel capacity:

$$\lim_{j\to\infty} I(\tilde{X}^{(j)}; \beta\tilde{X}^{(j)} + Z) \to \psi(\beta^2 P), \quad \tilde{X}^{(j)} \sim \mu^{(j)}. \tag{38}$$

One choice of such sequence of distributions has been explored in [13, Section VII]. In particular, $\mu^{(j)}$ is chosen to be a scaled and centered version of $\mathrm{Binomial}(j-1, 1/2)$ distribution. The convergence (38) follows from the asymptotic normality of binomial distributions.

For non-integer 2^{R_0}, time-sharing is applied similarly as in Section III-B. Pick $k_1, k_2, n_1, n_2 \in \mathbb{N}$ such that R' (defined in (22)) is arbitrarily close to R_0. Consider (n_1+n_2) transmission blocks of blocklength n each. For $i = 1, 2$, the $(\tilde{\Delta}\mathbb{Z})^n$-supported $(n, \tilde{R}, \tilde{P})$-Gaussian code $\tilde{\mathcal{C}}$ is superposed with the modulo quantization scheme with k_i messages and $\Delta_i = \tilde{\Delta}/k_i$ for n_i transmission blocks. The average transmit power is still upper bounded by P, and the overall average error probability is still $P_e(\tilde{C})$. The time-shared relay rate is $R' \leq R_0$ while the overall achieved rate is $\tilde{R} + R'$, which can be made as close as possible to $\tilde{R} + R_0$. Note that the achievable rate depends linearly on the relay link capacity, i.e., the former is the sum of the latter and the Gaussian code rate, therefore no loss is incurred by time-sharing.

The results of this subsection is summarized as follows.

Theorem 3. *For the deterministic Gaussian primitive relay channel depicted in Fig. 1(c), the channel capacity $\psi(\beta^2 P) + R_0$ is achievable using superposed modulo quantization coding under any average transmit power constraint $P > 0$.*

IV. Discussions and Future Work

This paper proposes a practical scalar modulo quantization coding scheme for the Gaussian primitive relay channel with perfectly correlated noises. When the noiseless relay-to-receiver link has capacity R_0, the modulo quantization scheme is able to achieve an overall rate of R_0 without error. By superposing the modulo quantization code onto a Gaussian code supported on a scaled integer lattice, the full capacity of this relay channel can be achieved.

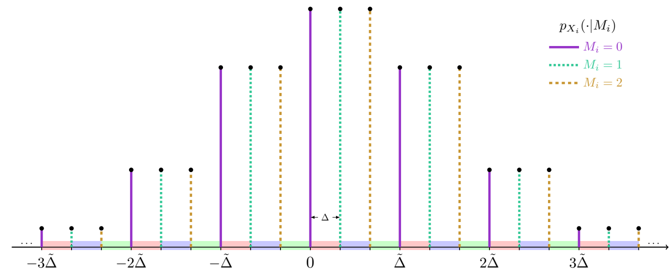

Fig. 4. Illustration of the X_i-marginal input distribution $p_{X_i}(\cdot|M_i)$ of the codebook (27) conditioned on the modulo quantization message $M_i \in \{0, 1, \cdots, k-1\}$. (Here, $p_{X_i}(\cdot|M^n) = p_{X_i}(\cdot|M_i)$, $i = 1, \cdots, n$.) The relay rate is $R_0 = \log 3$, i.e., $k = 3$ and $\Delta = \tilde{\Delta}/3$. In the decoding process, after M_i is used to subtract the correlated noise, we obtain a shifted codebook whose X_i-marginals are plotted in purple (solid line). As $\tilde{\Delta} \to 0$, this marginal distribution approaches the centered Gaussian distribution with variance P, which is capacity achieving for the direct channel.

The proposed modulo quantization scheme is not restricted to the relay channel with Gaussian noise. The proof of achievability depends only on the additive-noise structure of the channel and the fact that the noises are identical. For a primitive relay channel with generic additive noises that are perfectly correlated, the superposed modulo quantization scheme can also achieve channel capacity as long as there exists a sequence of capacity-achieving lattice-supported codes for the direct channel.

The proposed scheme resembles the HF scheme discussed earlier, which is only applicable to discrete-alphabet channels. The proposed modulo quantization scheme can be viewed as a modification of the hash-and-forward scheme, as the quantization process converts the continuous-alphabet channel to a discrete-alphabet channel, and then the modulo operation is used as a structured hashing method. We show in this paper that when perfect noise correlation is present, this modified HF scheme is able to achieve the capacity of the channel in Fig. 1(c).

The proposed modulo quantization scheme in Section III-B is related to a previously proposed color-and-forward scheme [14]. Color-and-forward is a coding strategy for discrete-alphabet primitive relay channels based on graph coloring that ensures zero error probability. While it may be possible to interpret the modulo quantization strategy proposed in this paper as a generalized form of coloring, the focus of this paper is quite different from that of [14]. Our aim is on how to take advantage of the noise correlation for a continuous-alphabet channel and how to superpose an error-correcting code for the Gaussian channel onto a modulo quantization scheme.

While most of the discussions in this paper are restricted to the case where the noises are perfectly correlated, we comment here that the modulo quantization coding scheme can still be applied when the noise correlation is not perfect. In this case, the decoding would not be error free; a channel code would need to be applied on top of the modulo quantization scheme. This is left as future work.

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