CONSTANT-MODULUS LINEAR TRANSFORM FOR RIS BEAMFORMING IN UPLINK MULTIUSER MIMO SYSTEMS

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ABSTRACT

A reconfigurable intelligent surface (RIS) with a large number of reflecting elements is capable of highly directed focusing for multiuser wireless cellular service provisioning, but to fully reap the benefit of RIS, an efficient beamforming algorithm is required. This paper investigates an RIS-assisted uplink multiuser multiple-input multiple-output (MU-MIMO) system and proposes a low-complexity algorithm for a max-min fairness beamforming problem. This paper identifies the problem as a max-min fractional program with constant-modulus constraints. To solve the problem efficiently, a novel constant-modulus linear transform is proposed to deal with the RIS-specific fractional functions. In this way, the max-min problem is transformed into a sequence of sub-problems, each of which can be solved to global optimality with a low complexity, provided that a specific condition is satisfied, thereby making the overall algorithm highly efficient. Numerical results demonstrate the effectiveness and the efficiency of the proposed algorithm.

***Index Terms*—** Reconfigurable intelligent surface, uplink, max-min fractional programming, constant-modulus linear transform.

1. INTRODUCTION

Future wireless networks are expected to provide ultra-high spectrum and energy efficiency, as well as ubiquitous coverage and reliable connectivity. To help realize these ambitious goals, the reconfigurable intelligent surface (RIS) has been proposed as a promising technology to facilitate the control of the wireless propagation environment in a programmable manner. Specifically, an RIS is a low-power planar structure equipped with a large number of reflecting elements that can alter the phases of incident electromagnetic (EM) waves. With the assistance of RIS, the wireless environment becomes part of the network design parameters that are subject to optimizations. This paper studies an uplink multiuser multiple-input multiple-output (MU-MIMO) communication system assisted by an RIS. The focus of this paper is on the max-min signal-to-interference-plus-noise-ratio (SINR) reflection beamforming optimization for the RIS that ensures fairness among the users.

There exist many prior works investigating fair beamforming designs for MU-MIMO systems with and without the assistance of RISs. For example, [1] investigates the max-min fairness beamforming design problem for a MU-MIMO interference channel and proposes an iterative algorithm based on solving a sequence of second order cone programming (SOCP) feasibility problems, which is computationally expensive. In [2–7], the integration of RIS into MIMO systems is explored to help improve performance fairness. In [2–4], the antenna beamformer and the RIS beamformer are alternately optimized, and the bisection approach and the semidefinite relaxation (SDR) technique are employed to maximize the minimum-SINR. In [5, 6], the authors respectively investigate multicell and cell-free MIMO systems, while applying the SDR to optimize the RIS beamformer. In addition, [7] investigates the same objective in RIS-assisted MIMO vehicular networks, where SDR is also employed. All these works rely on algorithms with high computational complexity, which limit the practicality in real-time or large-scale deployments, especially when the RIS has a large number of reflecting elements.

The main focus of this paper is to develop algorithms with low complexity to optimize RIS coefficients, which is essential for the practical implementation of RIS-assisted MU-MIMO systems. Specifically, we tackle a problem of maximizing the minimum uplink SINR using the ideas of fractional programming [8] and constant-modulus linear transform [9, 10] to transform a fractional function over constant-modulus variables to a linear function. In this way, the max-min SINR beamforming problem can be transformed into a sequence of sub-problems. We show that each sub-problem can be solved globally optimally with low complexity (provided that a specific condition is satisfied), thereby making the overall iterative optimization process highly efficient.

2. SYSTEM MODEL AND PROBLEM FORMULATION

Consider an uplink RIS-assisted MU-MIMO system, where an RIS consisting of N reflecting elements is used to enhance the communication between an M-antenna base station and K single-antenna users. The reflection coefficients of the RIS are denoted by

$$\mathbf{x} \triangleq \left[\exp\left(j\theta_1\right), \ldots, \exp\left(j\theta_N\right)\right]^{\mathsf{T}}, \tag{1}$$

where $\theta_n \in [-\pi, \pi)$ denotes the phase shift of the n-th element of the RIS. In this model of RIS beamforming, the optimizing variable $\mathbf{x}$ has a unit-modulus constraint, $|x_n| = 1, \forall n$, which is a challenging constraint to deal with. Further, the RIS beamformer is coupled with the BS receive combiners.

The channel from the RIS to the BS is represented as $\mathbf{G}$, the channel from the k-th user to the RIS is represented as $\mathbf{h}_k$, and the channel from the k-th user to the BS is represented as $\mathbf{d}_k$. The cascaded channel of the k-th user can be expressed as $\mathbf{G}\,\mathrm{diag}\{\mathbf{h}_k\}\mathbf{x} \triangleq \mathbf{H}_k\mathbf{x}$. The channel matrices $\mathbf{d}_k$'s and $\mathbf{H}_k$'s are assumed to be known at the BS by virtue of some channel estimation methods; see, e.g., [11] and the references therein.

Assume that the received signal $\mathbf{y}$ at the BS is processed by the receive combining matrix $\mathbf{W}$ with the k-th unit-norm column vector designed for the k-th user. Then, the output of the receive combiner for the k-th user is expressed as follows:

$$\hat{s}_k = \mathbf{w}_k^{\mathsf{H}}\mathbf{y} = \sum_{i=1}^{K} \mathbf{w}_k^{\mathsf{H}}\left(\mathbf{d}_i + \mathbf{H}_i\mathbf{x}\right)s_i + \mathbf{w}_k^{\mathsf{H}}\mathbf{n}, \tag{2}$$

where s_k represents the signal transmitted by the k-th user, and $\mathbf{n}$ represents the additive noise distributed as $\mathcal{CN}(\mathbf{0}, \sigma_n^2\mathbf{I})$. The signals transmitted by the users are mutually independent and distributed as $\mathcal{CN}(0, p_k)$ with transmit power p_k.

Given an RIS beamformer $\mathbf{x}$, optimizing the receive combiner $\mathbf{w}_k$ to maximize the SINR is a generalized Rayleigh quotient problem. The optimal $\mathbf{w}_k^\star$ can be given in closed-form as follows:

$$\mathbf{w}_k^\star = \frac{(\mathbf{R}_k(\mathbf{x}))^{-1}(\mathbf{d}_k + \mathbf{H}_k\mathbf{x})}{\left\|(\mathbf{R}_k(\mathbf{x}))^{-1}(\mathbf{d}_k + \mathbf{H}_k\mathbf{x})\right\|_2}, \tag{3}$$

where the matrix $\mathbf{R}_k(\mathbf{x})$ denotes the covariance matrix of the interference and noise and is given by

$$\mathbf{R}_k(\mathbf{x}) = \sum_{i\neq k}^{K} p_i\,(\mathbf{d}_i + \mathbf{H}_i\mathbf{x})(\mathbf{d}_i + \mathbf{H}_i\mathbf{x})^{\mathsf{H}} + \sigma_n^2\mathbf{I}. \tag{4}$$

With the optimal $\mathbf{w}_k^\star$, the SINR of the k-th user is given by

$$\gamma_k(\mathbf{x}) = p_k\,(\mathbf{d}_k + \mathbf{H}_k\mathbf{x})^{\mathsf{H}}(\mathbf{R}_k(\mathbf{x}))^{-1}(\mathbf{d}_k + \mathbf{H}_k\mathbf{x}). \tag{5}$$

One can observe from (5) that the SINR is a multi-dimensional ratio function over the RIS reflection coefficients $\mathbf{x}$.

In this paper, we aim to provide a fair quality-of-service for all users via optimizing the RIS reflection coefficients $\mathbf{x}$, which can be formulated as a max-min problem as follows:

$$\textbf{(P1):}\quad \max_{\mathbf{x}}\ \min_{k}\ \ \gamma_k(\mathbf{x}) \tag{6a}$$

$$\text{subject to}\ \ |x_n| = 1,\ \forall n. \tag{6b}$$

Problem (P1) is a max-min-multi-dimensional-ratios problem with constant-modulus constraints. In the sequel, we propose a highly efficient algorithm to solve this problem.

3. LINEAR TRANSFORM FOR BEAMFORMER DESIGN

3.1. Constant-Modulus Linear Transform

To solve problem (P1) in an efficient manner, we employ a technique called the constant-modulus linear transform, which is first proposed in our prior works [9, 10], to tackle ratios for MIMO sensing. Here, we propose a novel way of utilizing this technique, specifically tailored to the fractional structures of problem (P1).

We begin by defining the following general form of the ratio function in order to later tackle problem (P1):

$$\mathsf{f}(\mathbf{x}) \triangleq (\mathbf{a} + \mathbf{A}\mathbf{x})^{\mathsf{H}}(\mathbf{\Sigma}(\mathbf{x}))^{-1}(\mathbf{a} + \mathbf{A}\mathbf{x}), \tag{7}$$

where the variable $\mathbf{x}$ is an N-dimensional complex vector with each entry being unit-modulus, the denominator matrix $\mathbf{\Sigma}(\mathbf{x})$ is

$$\mathbf{\Sigma}(\mathbf{x}) \triangleq \sum_r (\mathbf{b}_r + \mathbf{B}_r\mathbf{x})(\mathbf{b}_r + \mathbf{B}_r\mathbf{x})^{\mathsf{H}} + \mathbf{C}, \tag{8}$$

and the constant matrix $\mathbf{C}$ is positive definite.

To establish the constant-modulus linear transform for $\mathsf{f}(\mathbf{x})$, we first apply the quadratic transform technique [8]. The result is given in the following lemma.

Lemma 1. *A lower bound for $\mathsf{f}(\mathbf{x})$ is the following:*

$$\mathsf{f}(\mathbf{x}) \geq 2\,\Re\mathfrak{e}\Big\{(\mathbf{a} + \mathbf{A}\mathbf{x})^{\mathsf{H}}\boldsymbol{\lambda}\Big\} - \boldsymbol{\lambda}^{\mathsf{H}}\mathbf{\Sigma}(\mathbf{x})\boldsymbol{\lambda},\ \ \forall\,\mathbf{x}, \boldsymbol{\lambda}, \tag{9}$$

with the equality achieved at

$$\boldsymbol{\lambda}^\star = (\mathbf{\Sigma}(\mathbf{x}))^{-1}(\mathbf{a} + \mathbf{A}\mathbf{x}). \tag{10}$$

After some algebraic manipulation, we can rewrite the quadratic term in (9) as follows:

$$\boldsymbol{\lambda}^{\mathsf{H}}\mathbf{\Sigma}(\mathbf{x})\boldsymbol{\lambda} = \mathbf{x}^{\mathsf{H}}\mathbf{M}(\boldsymbol{\lambda})\mathbf{x} + 2\,\Re\mathfrak{e}\Big\{\mathbf{x}^{\mathsf{H}}\mathbf{v}(\boldsymbol{\lambda})\Big\} + c(\boldsymbol{\lambda}), \tag{11}$$

where the matrix $\mathbf{M}(\boldsymbol{\lambda})$ is given by

$$\mathbf{M}(\boldsymbol{\lambda}) = \sum_r \left(\mathbf{B}_r^{\mathsf{H}}\boldsymbol{\lambda}\right)\left(\mathbf{B}_r^{\mathsf{H}}\boldsymbol{\lambda}\right)^{\mathsf{H}}, \tag{12}$$

the vector $\mathbf{v}(\boldsymbol{\lambda})$ is given by

$$\mathbf{v}(\boldsymbol{\lambda}) = \sum_r \left(\mathbf{B}_r^{\mathsf{H}}\boldsymbol{\lambda}\right)\left(\mathbf{b}_r^{\mathsf{H}}\boldsymbol{\lambda}\right)^{\mathsf{H}}, \tag{13}$$

and $c(\boldsymbol{\lambda})$ is given by

$$c(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^{\mathsf{H}}\left(\sum_r \mathbf{b}_r\mathbf{b}_r^{\mathsf{H}} + \mathbf{C}\right)\boldsymbol{\lambda}. \tag{14}$$

We now eliminate the quadratic term $\mathbf{x}^{\mathsf{H}}\mathbf{M}(\boldsymbol{\lambda})\mathbf{x}$ in (11) by using a majorization-maximization technique [12, Eq. (26)], which is

$$\mathbf{x}^{\mathsf{H}}\mathbf{M}\mathbf{x} \leq \mathbf{x}^{\mathsf{H}}\mathbf{L}\mathbf{x} + \mathbf{z}^{\mathsf{H}}(\mathbf{L} - \mathbf{M})\,\mathbf{z} + 2\,\Re\mathfrak{e}\Big\{\mathbf{x}^{\mathsf{H}}(\mathbf{M} - \mathbf{L})\,\mathbf{z}\Big\}, \tag{15}$$

where $\mathbf{L} \succeq \mathbf{M}$, and the equality is achieved at $\mathbf{z} = \mathbf{x}$. Then, by replacing $\mathbf{L}$ with $\delta\mathbf{I}$, where δ is the trace of $\mathbf{M}(\boldsymbol{\lambda})$ so that $\delta\mathbf{I} \succeq \mathbf{M}(\boldsymbol{\lambda})$, and by combining with the fact that for unit-modulus $\mathbf{x}$ and $\mathbf{z}$,

$$\mathbf{x}^{\mathsf{H}}(\delta\mathbf{I})\,\mathbf{x} = \mathbf{z}^{\mathsf{H}}(\delta\mathbf{I})\,\mathbf{z} = \delta N, \tag{16}$$

we obtain the following constant-modulus linear transform for the multi-dimensional-ratio function $\mathsf{f}(\mathbf{x})$.

Theorem 1. *A lower bound for the multi-dimensional-ratio function $\mathsf{f}(\mathbf{x})$ can be constructed as follows:*

$$\mathsf{f}(\mathbf{x}) \geq 2\,\Re\mathfrak{e}\Big\{\mathbf{x}^{\mathsf{H}}\Big((\delta\mathbf{I} - \mathbf{M}(\boldsymbol{\lambda}))\,\mathbf{z} + \left(\mathbf{A}^{\mathsf{H}}\boldsymbol{\lambda} - \mathbf{v}(\boldsymbol{\lambda})\right)\Big)\Big\} + \bar{c}(\mathbf{z}, \boldsymbol{\lambda}) \triangleq \bar{\mathsf{f}}(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}),\quad \forall\,\boldsymbol{\lambda},\ \forall\,\mathbf{x}, \mathbf{z} \in \mathbb{S}^N, \tag{17}$$

where $\mathbb{S} = \{x \in \mathbb{C} \mid |x| = 1\}$, and $\bar{c}(\mathbf{z}, \boldsymbol{\lambda})$ is given by

$$\bar{c}(\mathbf{z}, \boldsymbol{\lambda}) = 2\,\Re\mathfrak{e}\Big\{\mathbf{a}^{\mathsf{H}}\boldsymbol{\lambda}\Big\} + \mathbf{z}^{\mathsf{H}}\mathbf{M}(\boldsymbol{\lambda})\mathbf{z} - 2\delta N - c(\boldsymbol{\lambda}). \tag{18}$$

The equality in (17) *is achieved at*

$$\mathbf{z}^\star = \mathbf{x}, \tag{19}$$

$$\boldsymbol{\lambda}^\star = (\mathbf{\Sigma}(\mathbf{x}))^{-1}(\mathbf{a} + \mathbf{A}\mathbf{x}). \tag{20}$$

Theorem 1 directly provides an equivalent reformulation of optimization problems involving ratio functions. Specifically, the max-min-ratios problem can be reformulated in the following way.

Corollary 1. *The max-min-multi-dimensional-ratios problem with constant-modulus constraints*

$$\max_{\mathbf{x}}\ \min_{i}\ \ w_i\mathsf{f}_i(\mathbf{x}) \tag{21a}$$

$$\text{subject to}\ \ |x_n| = 1,\ \forall n, \tag{21b}$$

where $w_i > 0$, $\forall i$, is equivalent to

$$\max_{\mathbf{x},\mathbf{z},\boldsymbol{\lambda}}\ \min_{i}\ \ w_i\bar{\mathsf{f}}_i(\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}_i) \tag{22a}$$

$$\text{subject to}\ \ |x_n| = |z_n| = 1,\ \forall n. \tag{22b}$$

3.2. Max-Min SINR Beamforming

We now solve the max-min SINR beamforming problem by utilizing the proposed linear transform in the following steps, which would eventually lead to a low-complexity solution.

Applying Corollary 1, problem (P1) can be transformed into the following equivalent problem:

$$\textbf{(P2):} \quad \max_{\mathbf{x},\mathbf{z},\boldsymbol{\lambda}} \ \min_{k} \ p_k \bar{\gamma}_k(\mathbf{x},\mathbf{z},\boldsymbol{\lambda}_k) \tag{23a}$$

$$\text{subject to} \ \ |x_n| = |z_n| = 1, \ \forall n. \tag{23b}$$

Here, the new transformed function $\bar{\gamma}_k(\mathbf{x},\mathbf{z},\boldsymbol{\lambda}_k)$ is given by

$$\bar{\gamma}_k(\mathbf{x},\mathbf{z},\boldsymbol{\lambda}_k) = \bar{c}_k(\mathbf{z},\boldsymbol{\lambda}_k) + \\ 2\,\mathfrak{Re}\Big\{\mathbf{x}^{\mathsf{H}}\left(\left(\delta_k\mathbf{I} - \bar{\mathbf{M}}_k(\boldsymbol{\lambda}_k)\right)\mathbf{z} + \left(\mathbf{H}_k^{\mathsf{H}}\boldsymbol{\lambda}_k - \bar{\mathbf{v}}_k(\boldsymbol{\lambda}_k)\right)\right)\Big\}, \tag{24}$$

where the matrix $\bar{\mathbf{M}}_k(\boldsymbol{\lambda}_k)$ is given by

$$\bar{\mathbf{M}}_k(\boldsymbol{\lambda}_k) = \sum_{i\neq k} p_i \left(\mathbf{H}_i^{\mathsf{H}}\boldsymbol{\lambda}_k\right)\left(\mathbf{H}_i^{\mathsf{H}}\boldsymbol{\lambda}_k\right)^{\mathsf{H}}, \tag{25}$$

the parameter δ_k is the trace of $\bar{\mathbf{M}}_k(\boldsymbol{\lambda}_k)$, $\bar{\mathbf{v}}_k(\boldsymbol{\lambda}_k)$ is given by

$$\bar{\mathbf{v}}_k(\boldsymbol{\lambda}_k) = \sum_{i\neq k} p_i \left(\mathbf{H}_i^{\mathsf{H}}\boldsymbol{\lambda}_k\right)\left(\mathbf{d}_i^{\mathsf{H}}\boldsymbol{\lambda}_k\right)^{\mathsf{H}}, \tag{26}$$

and $\bar{c}_k(\mathbf{z},\boldsymbol{\lambda}_k)$ is given by

$$\bar{c}_k(\mathbf{z},\boldsymbol{\lambda}_k) = 2\,\mathfrak{Re}\Big\{\mathbf{d}_k^{\mathsf{H}}\boldsymbol{\lambda}_k\Big\} + \mathbf{z}^{\mathsf{H}}\bar{\mathbf{M}}_k(\boldsymbol{\lambda}_k)\mathbf{z} - 2\,\delta_k N \\ - \boldsymbol{\lambda}_k^{\mathsf{H}}\left(\sum_{i\neq k}\mathbf{d}_i\mathbf{d}_i^{\mathsf{H}} + \sigma_n^2\mathbf{I}\right)\boldsymbol{\lambda}_k. \tag{27}$$

We now solve problem (P2) in an iterative manner. When $\mathbf{x}$ is fixed, the optimal $\mathbf{z}^\star$ and $\boldsymbol{\lambda}_k^\star$ are given in closed-form by

$$\mathbf{z}^\star = \mathbf{x}, \tag{28}$$

$$\boldsymbol{\lambda}_k^\star = \left(\mathbf{R}_k(\mathbf{x})\right)^{-1}\left(\mathbf{d}_k + \mathbf{H}_k\mathbf{x}\right). \tag{29}$$

When $\mathbf{z}$ and $\boldsymbol{\lambda}$ are held fixed, updating $\mathbf{x}$ requires us to solve the following problem:

$$\max_{\mathbf{x}} \ \min_{k} \ p_k\bar{\gamma}_k(\mathbf{x},\mathbf{z},\boldsymbol{\lambda}_k) \tag{30a}$$

$$\text{subject to} \ \ |x_n| = 1, \ \forall n, \tag{30b}$$

which is equivalent to

$$\max_{\mathbf{x}} \ \min_{\boldsymbol{\nu}} \ \mathfrak{L}(\mathbf{x},\boldsymbol{\nu}) \triangleq \sum_k \nu_k\, p_k\, \bar{\gamma}_k(\mathbf{x},\mathbf{z},\boldsymbol{\lambda}_k) \tag{31a}$$

$$\text{subject to} \ \ |x_n| = 1, \ \forall n, \tag{31b}$$

$$\sum_k \nu_k = 1, \ \nu_k \geq 0, \ \forall k. \tag{31c}$$

Since the constraint (31b) is non-convex and the problem (31) is of max-min form, solving it is still challenging. To solve the problem efficiently, we first relax the constraint (31b) into $|x_n| \leq 1, \forall n$. The relaxed version of the problem (31) is given by

$$\max_{\mathbf{x}} \ \min_{\boldsymbol{\nu}} \ \mathfrak{L}(\mathbf{x},\boldsymbol{\nu}) \tag{32a}$$

$$\text{subject to} \ \ |x_n| \leq 1, \ \forall n, \tag{32b}$$

$$\sum_k \nu_k = 1, \ \nu_k \geq 0, \ \forall k. \tag{32c}$$

It is clear that the optimal solution of the relaxed problem (32) is an upper bound to that of problem (31), because the relaxed problem (32) has a larger feasible set than the problem (31).

3.3. Low-Complexity Algorithm

Observe that the objective function (32a) is linear (i.e., both convex and concave) over $\mathbf{x}$ and $\boldsymbol{\nu}$, the constraints are convex, and the domains are compact, so we can interchange min and max, i.e., the optimal objective value of the max-min problem (32) is identical to the optimal objective value of the following min-max problem:

$$\min_{\boldsymbol{\nu}} \ \max_{\mathbf{x}} \ \mathfrak{L}(\mathbf{x},\boldsymbol{\nu}) \tag{33a}$$

$$\text{subject to} \ \ |x_n| \leq 1, \ \forall n, \tag{33b}$$

$$\sum_k \nu_k = 1, \ \nu_k \geq 0, \ \forall k. \tag{33c}$$

Now, we make two key observations. First, for fixed $\boldsymbol{\nu}$, the inner maximization in (33a) can be solved by setting the elements of $\mathbf{x}$ to match the phases of the linear coefficient in (33a) as follows:

$$\mathbf{x}(\boldsymbol{\nu}) = \exp\left[j\arg\left(\boldsymbol{\varsigma}(\boldsymbol{\nu})\right)\right], \tag{34}$$

where $\boldsymbol{\varsigma}(\boldsymbol{\nu})$ denotes the overall linear coefficient, i.e.,

$$\boldsymbol{\varsigma}(\boldsymbol{\nu}) = \sum_{k=1}^{K}\nu_k\, p_k\left(\left(\delta_k\mathbf{I} - \bar{\mathbf{M}}_k(\boldsymbol{\lambda}_k)\right)\mathbf{z} + \left(\mathbf{H}_k^{\mathsf{H}}\boldsymbol{\lambda}_k - \bar{\mathbf{v}}_k(\boldsymbol{\lambda}_k)\right)\right). \tag{35}$$

This gives a highly efficient procedure for optimizing $\mathbf{x}$. Note that the above method (i.e., using (34) to find the optimal $\mathbf{x}$) only works when $\boldsymbol{\varsigma}(\boldsymbol{\nu})$ is non-zero in every component. For now, let us assume that this is true.

It remains to find the optimal $\boldsymbol{\nu}$. Substituting $\mathbf{x}(\boldsymbol{\nu})$ in (34) into (33a), the problem (33) can be rewritten as

$$\textbf{(P3):} \quad \underset{\boldsymbol{\nu}}{\text{minimize}} \ \ \mathfrak{L}(\mathbf{x}(\boldsymbol{\nu}),\boldsymbol{\nu}) \tag{36a}$$

$$\text{subject to} \ \ \sum_k \nu_k = 1, \ \nu_k \geq 0, \ \forall k, \tag{36b}$$

where the objective function is given by

$$\mathfrak{L}(\mathbf{x}(\boldsymbol{\nu}),\boldsymbol{\nu}) = 2\left\|\boldsymbol{\varsigma}(\boldsymbol{\nu})\right\|_1 + \sum_{k=1}^{K}\nu_k\, p_k\, \bar{c}_k(\mathbf{z},\boldsymbol{\lambda}_k). \tag{37}$$

Our second key observation is that this is a linear program in which the dimension of the variable $\boldsymbol{\nu}$ is $K \ll N$. Thus, this approach of first obtaining the optimal $\boldsymbol{\nu}^\star$ by solving (P3), then obtaining $\mathbf{x}(\boldsymbol{\nu}^\star)$ based on (34) can be used to solve the problem (33) very efficiently.

Let $\boldsymbol{\nu}^\star$ denote an optimal solution of problem (P3). As already mentioned, when none of the entries of $\boldsymbol{\varsigma}(\boldsymbol{\nu}^\star)$ are zero, i.e.,

$$\left[\boldsymbol{\varsigma}(\boldsymbol{\nu}^\star)\right]_n \neq 0, \ \forall n, \tag{38}$$

$\mathbf{x}(\boldsymbol{\nu}^\star)$ obtained from (34) would be a *unique* maximizer of $\mathfrak{L}(\mathbf{x},\boldsymbol{\nu}^\star)$.

We now justify that this $\mathbf{x}(\boldsymbol{\nu}^\star)$ is also the solution to the original max-min problem (30) if the condition (38) holds. Let $(\bar{\mathbf{x}},\bar{\boldsymbol{\nu}})$ be any optimal solution of the max-min problem (32). Then, we have the following relations:

$$\mathsf{d}^\star = \mathfrak{L}(\mathbf{x}(\boldsymbol{\nu}^\star),\boldsymbol{\nu}^\star) \geq \mathfrak{L}(\bar{\mathbf{x}},\boldsymbol{\nu}^\star) \geq \mathfrak{L}(\bar{\mathbf{x}},\bar{\boldsymbol{\nu}}) = \mathsf{p}^\star, \tag{39}$$

where $\mathsf{p}^\star$ and $\mathsf{d}^\star$ represent the optimal objective values of the problems (32) and (33), respectively. Since $\mathsf{d}^\star = \mathsf{p}^\star$, we have

$$\mathfrak{L}(\mathbf{x}(\boldsymbol{\nu}^\star),\boldsymbol{\nu}^\star) = \mathfrak{L}(\bar{\mathbf{x}},\boldsymbol{\nu}^\star). \tag{40}$$

Therefore, under the condition (38), $\mathbf{x}(\boldsymbol{\nu}^\star) = \bar{\mathbf{x}}$ is the optimal solution of the relaxed problem (32). Since $\mathbf{x}(\boldsymbol{\nu}^\star)$ also satisfies the constant-modulus constraint, it must be an optimal solution of the problem (31), and hence an optimal solution of (30). We summarize the above result in the following theorem.

Theorem 2. *If the optimal solution $\boldsymbol{\nu}^\star$ to problem (P3) satisfies the condition* (38)*, then $\mathbf{x}(\boldsymbol{\nu}^\star)$ is the optimal solution to the non-convex max-min problem* (30)*.*

Empirically, for the scenario considered in this paper, where the number of RIS reflecting elements is much greater than the number of users, i.e., $N \gg K$, (38) is almost always satisfied. It can also be shown that if the condition (38) is satisfied at every iteration of the proposed algorithm, the algorithm always converges to a *stationary point* of the original problem (P1).

4. NUMERICAL RESULTS

In this section, we provide simulation results to show the efficiency and the effectiveness of the proposed algorithm. The simulation environment is given as follows:

- The RIS is a planar with 15^2 or 20^2 reflecting elements, and the BS is equipped with 8 antennas.
- The distance between the BS and the RIS is 100 meters with pathloss exponent 2, the distance between the users and the RIS is 20 meters with pathloss exponent 2, and the distance between the users and the BS is approximately 100 meters with pathloss exponent 3.5.
- The BS-RIS and BS-users channels are modeled as Rayleigh fading. The RIS-user channel is modeled as Rician fading with Rician factor of 10. The noise power is set as -80 dBm.

We first show that the designed beamforming solutions are interpretable. To illustrate this, we plot the RIS beampattern, which is the power of the signals received at the BS from a unit-power signal transmitted in different azimuth angles with respect to the RIS. From Fig. 1, one can observe that the solutions produce beams that are aligned well with the directions of users in both cases of three users and five users, with relatively fair beam quality among users.

To evaluate the efficiency of the proposed algorithm, we compare it with several benchmarks. In most prior works, e.g., [2–7], the receive combiner and the RIS beamformer are alternately optimized. In each iteration, the bisection method, along with the SDR technique are used to find the maximal minimum-SINR. This baseline is referred to as Bi-SDR. Alternatively, it is possible to optimize the RIS by first quantizing the phase shifts into 8-bit levels, followed by using the coordinate ascent algorithm, referred to as Di-CoD.

Table 1 shows the SINR performance and the run-time of the proposed algorithm, referred to as CM-LT, and the benchmarks under the same setup. It can be observed that the proposed algorithm not only achieves the highest SINR values but also is less complex, particularly as N increases. Note that CM-LT can be further accelerated via the momentum method [13].

To further illustrate the performance, we plot the minimum rate among the users versus the transmit power utilizing different algorithms in Fig. 2. It can be observed that the proposed algorithm outperforms the benchmarks. Furthermore, the Bi-SDR method suffers from significantly higher complexity. Moreover, quantizing the solution obtained by the proposed algorithm into 8-bit discrete phase shifts results in only a negligible performance loss. In contrast, directly applying the coordinate ascent algorithm in the discrete phase shifts leads to significantly inferior performance.

5. CONCLUSION

This paper proposes an efficient method for max-min-SINR beamforming design for an RIS-assisted uplink MU-MIMO system. The problem is a max-min fractional program with constant-modulus constraints. We propose a novel constant-modulus linear transform technique that transforms the max-min problem into a sequence of subproblems, each of which can be optimally solved with low complexity in terms of the number of RIS reflecting elements.

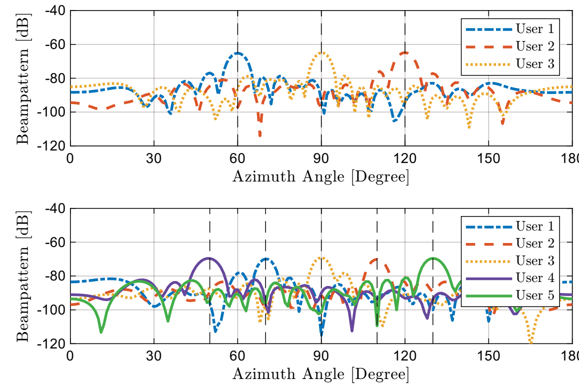

Fig. 1. RIS reflecting beampatterns with the max-min-SINR beamformers (black dashed lines indicate the true user directions).

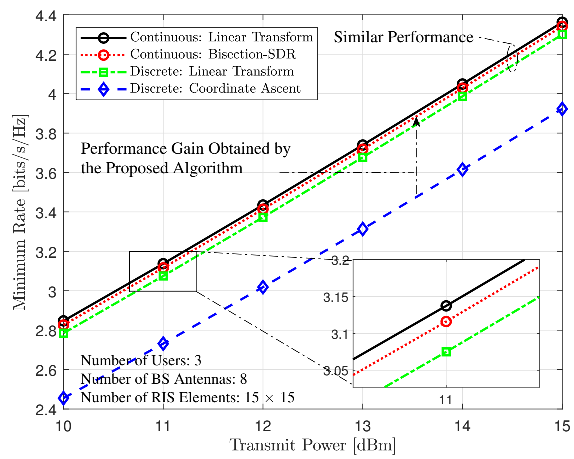

Fig. 2. Minimum rate vs. transmit power with different algorithms.

Table 1. Optimized value and run-time for max-min SINR.

	Minimum SINR	Convergence Iteration No.	Convergence Run-time [s]
	$K=3,\ N=15\times 15=225$		
CM-LT	19.57	767	89.1
Bi-SDR	18.34	54	362.6
Di-CoD	14.16	58	307.9
	$K=3,\ N=20\times 20=400$		
CM-LT	50.06	1183	152.1
Bi-SDR	49.24	82	3675.4
Di-CoD	32.83	45	746.9
	$K=5,\ N=20\times 20=400$		
CM-LT	23.49	2142	330.8
Bi-SDR	22.17	46	3533.5
Di-CoD	17.12	50	1337.9

6. REFERENCES

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