Sum Capacity of Gaussian Vector Broadcast Channels

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Abstract

This paper characterizes the sum capacity of a class of potentially non-degraded Gaussian vector broadcast channels where a single transmitter with multiple transmit terminals sends independent information to multiple receivers. Coordination is allowed among the transmit terminals, but not among the receive terminals. The sum capacity is shown to be a saddle-point of a Gaussian mutual information game, where a signal player chooses a transmit covariance matrix to maximize the mutual information, and a fictitious noise player chooses a noise correlation to minimize the mutual information. The sum capacity is achieved using a precoding strategy for Gaussian channels with additive side information non-causally known at the transmitter. The optimal precoding structure is shown to correspond to a decision-feedback equalizer that decomposes the broadcast channel into a series of single-user channels with interference pre-subtracted at the transmitter.

I. Introduction

A communication situation where a single transmitter sends independent information to multiple uncoordinated receivers is referred to as a broadcast channel. Figure 1 illustrates a two-user broadcast channel, where independent messages W_1 and W_2 are jointly encoded by the transmitter X, and the receivers Y_1 and Y_2 are each responsible for decoding W_1 and W_2, respectively. A $(n, 2^{nR_1}, 2^{nR_2})$ codebook for a broadcast channel consists of an encoding function $X^n(W_1, W_2)$ where $W_1 \in \{1, \cdots, 2^{nR_1}\}$ and $W_2 \in \{1, \cdots, 2^{nR_2}\}$ and decoding functions $\hat{W}_1(Y_1^n)$ and $\hat{W}_2(Y_2^n)$. An error occurs when $W_1 \neq \hat{W}_1$ or $W_2 \neq \hat{W}_2$. A rate pair (R_1, R_2) is achievable if there exists a sequence of $(n, 2^{nR_1}, 2^{nR_2})$ codebooks for which the average probability of error $P_e^n \to 0$ as $n \to \infty$. The capacity region of a broadcast channel is the set of all achievable rate pairs.

The broadcast channel was first introduced by Cover [1], who also proposed an achievable coding strategy based on superposition. Superposition coding has been shown to be optimal for the class of degraded broadcast channels [2] [3]. However, it is in general sub-optimal for non-degraded broadcast channels. The largest achievable region for the non-degraded broadcast channel is due to Marton [4] [5], but no converse has been established, except in special cases such as deterministic broadcast channels and more capable broadcast channels. (See [6] for a comprehensive review.) The capacity region for the general broadcast channel is still an unsolved problem.

This paper makes progress on the broadcast channel problem by solving for the sum capacity of a particular class of non-degraded Gaussian vector broadcast channels. The main challenge in the broadcast channel problem is that a broadcast channel distributes information across several receivers, and without the joint processing of the received signals, it is not possible to communicate at a rate equal to the mutual information between the input and the outputs. The contribution of this paper is to show that for a Gaussian vector broadcast channel, an equivalent of receiver

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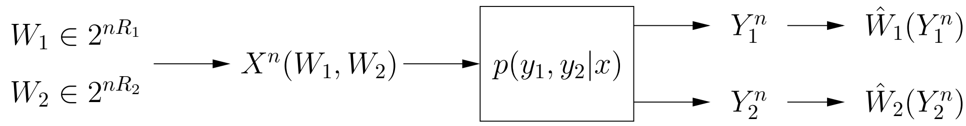


Fig. 1. Broadcast channel

processing can be implemented at the transmitter by precoding. Further, the optimal precoder takes the form of a generalized decision-feedback equalizer across the user domain. The solution to the sum capacity problem for the broadcast channel illustrates the value of cooperation at the receiver. Without receiver cooperation, the capacity of a Gaussian vector channel becomes a saddle-point of a mutual information game, where "nature" effectively puts forth a fictitious worst possible noise correlation.

The main result of this paper is a generalization of an earlier result by Caire and Shamai [7], who characterized the sum capacity of a broadcast channel with two users each equipped with a single antenna. The achievability proof of Caire and Shamai's result is based on a coding strategy called "writing on dirty paper" [8], and the converse is based on an upper bound by Sato [9]. This paper generalizes both the achievability and the converse to vector broadcast channels with an arbitrary number of transmit antennas and an arbitrary number of users each equipped with multiple receive antennas.

The sum capacity result has also been obtained in two simultaneous and independent work: [10] and [11]. These two separate work arrive at essentially the same result via a duality relation between the multiple access channel capacity region and the dirty-paper precoding region for the broadcast channel. The proof technique contained in this paper is quite different in that it reveals an equalization structure for the optimal broadcast strategy. This decision-feedback equalizer viewpoint leads directly to a path of implementation. It also makes the capacity result amenable to practical coding schemes, such as the inflated-lattice precoding strategy [12] and the trellis shaping technique [13].)

Further, the result in this paper is in fact more general than that of [10] and [11]. The result of this paper applies to broadcast channels with arbitrary convex input constraints, while the results of [10] and [11] appear to be applicable for broadcast channels with a total power constraint only. However, none of these work fully address the capacity region for the vector broadcast channel. The difficulty appears to be in proving that Gaussian inputs are optimal for non-rate-sum points. In fact, as is shown in [14] and [15], the dirty paper precoding region is the capacity region if an additional Gaussianity assumption is made. The capacity region of Gaussian vector broadcast channels is still an open problem.

The rest of this paper is organized as follows: In section II, the Gaussian vector broadcast channel problem is formulated, and a precoding scheme based on channels with transmitter side information is described. In section III, the optimal precoding structure is shown to be closely related to a generalized decision-feedback equalizer. In section IV, an outer bound for the sum capacity of the Gaussian broadcast channel is computed, and the decision-feedback precoder is shown to achieve the outer bound, thus proving the main capacity result. Section V summarizes the main result of the paper by illustrating the value of cooperation in a Gaussian vector channel.

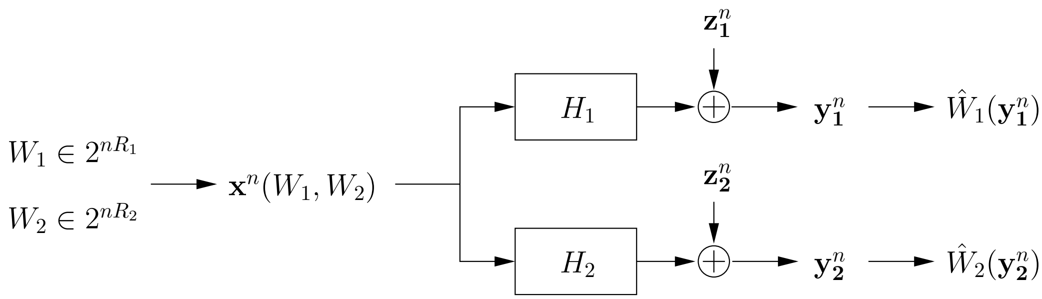


Fig. 2. Gaussian vector broadcast channel

The notations used in this paper are as follows. Lower case letters are used to denote scalars, e.g. x, y. Upper case letters are used to denote scalar random variables, e.g. X, Y, or matrices, e.g. H, where context should make the distinction clear. Bold face letters are used to denote vectors, e.g. $\mathbf{x}$, $\mathbf{y}$, or vector random variables, e.g. $\mathbf{X}$, $\mathbf{Y}$. For matrices, $\cdot^T$ denote the transpose operation and $|\cdot|$ denotes the determinant operation. The discussions in this paper are confined to the real-valued signals. However, all results extend naturally to the complex-valued case.

II. Precoding for Gaussian Broadcast Channels

A Gaussian vector broadcast channel refers to a broadcast channel where the law of the channel transition probability $p(y_1, y_2|x)$ is Gaussian, and where $\mathbf{x}$, $\mathbf{y}_1$ and $\mathbf{y}_2$ are vector valued. Figure 2 illustrates a two-user Gaussian vector broadcast channel:

$$\begin{aligned} \mathbf{y_1} &= H_1\mathbf{x} + \mathbf{z_1} \\ \mathbf{y_2} &= H_2\mathbf{x} + \mathbf{z_2}, \end{aligned} \tag{1}$$

where $\mathbf{x}$ is the transmit signal, $\mathbf{y_1}$ and $\mathbf{y_2}$ are receive signals, H_1, H_2 are channel matrices, and $\mathbf{z_1}$, $\mathbf{z_2}$ are Gaussian vector noises. Independent information is to be sent to each receiver. This paper characterizes the maximum sum rate $R_1 + R_2$. The development here is restricted to the two-user case for simplicity. The results can be generalized easily to channels with more than two users.

When a Gaussian broadcast channel has a scalar input and scalar outputs, it can be regarded as a degraded broadcast channel for which the capacity region is well established [16]. A broadcast channel is physically degraded if $p(\mathbf{y_1}, \mathbf{y_2}|\mathbf{x}) = p(\mathbf{y_1}|\mathbf{x})p(\mathbf{y_2}|\mathbf{y_1})$. Intuitively, this means that one user's signal is a noisier version of the other user's signal. Consider the Gaussian scalar broadcast channel:

$$\begin{aligned} y_1 &= x + z_1 \\ y_2 &= x + z_2, \end{aligned} \tag{2}$$

where x is the scalar transmitted signal subject to a power constraint P, y_1 and y_2 are the received signals, and z_1 and z_2 are the additive white Gaussian noises with variances σ_1^2 and σ_2^2, respectively. This broadcast channel is equivalent to a physically degraded channel for the following reason. Without loss of generality, assume $\sigma_1 < \sigma_2$. Then, z_2 can be re-written as $z_2' = z_1 + z'$, where $z' \sim \mathcal{N}(0, \sigma_2^2 - \sigma_1^2)$ is independent of z_1. Since z_2' has the same distribution as z_2, y_2 is now equivalent to $y_1 + z'$. Thus, y_2 can be regarded as a degraded version of y_1. The capacity region for a degraded broadcast channel is achieved using a superposition coding and interference subtraction scheme due

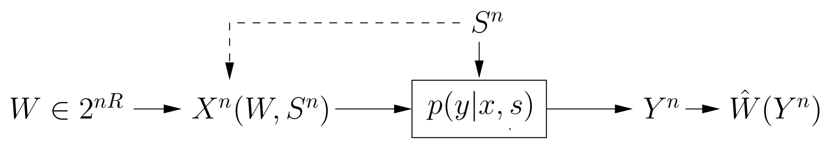


Fig. 3. Channel with non-causal transmitter side information

to Cover [1]. The idea is to divide the total power into $P_1 = \alpha P$ and $P_2 = (1-\alpha)P$ $(0 \leq \alpha \leq 1)$ and to construct two independent Gaussian codebooks for the two users with powers P_1 and P_2, respectively. To send two independent messages, one codeword is chosen from each codebook, and their sum is transmitted. Because y_2 is a degraded version of y_1, the codeword intended for y_2 can also be decoded by y_1. Thus, y_1 can subtract the effect of the codeword intended for y_2 and can effectively get a cleaner channel with noise power σ_1^2 instead of $\sigma_1^2 + P_2$. It is not difficult to see that the following rate pair is achievable:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2}\right) \tag{3}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2 + P_1}\right). \tag{4}$$

In fact, as was shown by Bergman [3], this superposition and interference subtraction scheme is optimal for the degraded Gaussian broadcast channel.

When a Gaussian broadcast channel has a vector input and vector outputs, it is no longer necessarily degraded, and superposition coding is no longer capacity-achieving. The capacity region for a non-degraded broadcast channel is still an unsolved problem. The largest achievable region in this case is due to Marton [4] [5], and it uses the idea of random binning. For a two-user broadcast channel with independent information for each user, the Marton's region is as follows:

$$R_1 \leq I(\mathbf{U_1}; \mathbf{Y_1}) \tag{5}$$

$$R_2 \leq I(\mathbf{U_2}; \mathbf{Y_2}) \tag{6}$$

$$R_1 + R_2 \leq I(\mathbf{U_1}; \mathbf{Y_1}) + I(\mathbf{U_2}; \mathbf{Y_2}) - I(\mathbf{U_1}; \mathbf{U_2}) \tag{7}$$

where $(\mathbf{U_1}, \mathbf{U_2})$ is a pair of auxiliary random variables, and the mutual information is evaluated under a joint distribution $p(\mathbf{x}|\mathbf{u_1}, \mathbf{u_2})p(\mathbf{u_1}, \mathbf{u_2})$ whose induced marginal distribution $p(\mathbf{x})$ satisfies the input constraint. Although the optimality of Marton's region is not known for the general broadcast channel, it is optimal for several classes of channels such the deterministic broadcast channel and more capable broadcast channels (see [6] and references herein.) The objective of this paper is to show that a proper choice of $(\mathbf{U_1}, \mathbf{U_2})$ also gives the sum-capacity of a non-degraded Gaussian vector broadcast channel.

As a first step, let's examine the degraded broadcast channel more carefully and give an interpretation of the auxiliary random variables in the degraded case. The connection between the degraded broadcast channel capacity region and Marton's region lies in the study of channels with non-causal transmitter side information. A channel with side information is illustrated in Figure 3. The channel output is a function of the input sequence X^n and a channel state sequence S^n. The channel state

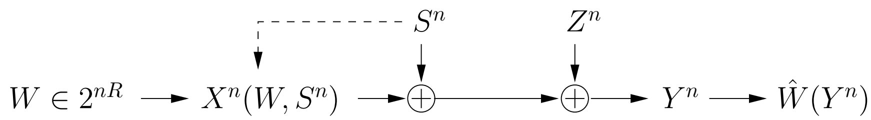


Fig. 4. Gaussian channel with transmitter side information

is not known to the receiver but is known to the transmitter as the side information. Further, the transmitter knows the entire state sequence S^n prior to transmission in a non-causal way. For such a channel, Gel'fand and Pinsker [17] and Heegard and El Gamal [18] showed that its capacity can be characterized using an auxiliary random variable U:

$$C = \max_{p(u,x|s)} \{I(U;Y) - I(U;S)\}. \tag{8}$$

The achievability proof of this result uses a random-binning argument, and it is closely connected to Marton's achievability region for the broadcast channel. Such a connection was noted by Gel'fand and Pinsker in [17], and was further used by Caire and Shamai [7] for the N-by-two Gaussian broadcast channel. The following argument illustrates the connection. Fix a pair of auxiliary random variables (U_1, U_2) and a conditional distribution $p(x|u_1, u_2)$. Consider the effective channel $p(y_1, y_2|x)p(x|u_1, u_2)$. Construct a random-coding codebook from U_2 to Y_2 using an i.i.d. distribution according to $p(u_2)$. Evidently, a rate of $R_2 = I(U_2; Y_2)$ is achievable. Now, since U_2 is completely known at the transmitter, the channel from U_1 to Y_1 is a channel with non-causal side information available at the transmitter. Then, Gel'fand and Pinsker's result ensures that a rate of $R_1 = I(U_1; Y_1) - I(U_2; U_1)$ is achievable. This rate pair is precisely a corner point in Marton's region for the broadcast channel. The above argument ignores the issue that U_1 now depends on U_2, but for the Gaussian channel, the argument can be made rigorous.

When specialized to the Gaussian channel, the capacity of a channel with side information has an interesting solution. Consider the Gaussian channel shown in Figure 4:

$$y = x + s + z, \tag{9}$$

where x and y are the transmitted and the received signals respectively, s is a Gaussian interfering signal whose entire non-causal realization is known to the transmitter but not to the receiver, and z is a Gaussian noise independent of s. In a surprising result known as "writing-on-dirty-paper," Costa [8] showed that when s and z are independent Gaussian random variables, under a fixed power constraint, the capacity of the channel with interference is the same as if the interference does not exist. In addition, the optimal transmit signal x is statistically independent of s. In effect, interference can be "pre-subtracted" at the transmitter without an increase in transmit power.

The "dirty-paper" result gives us another way to derive the degraded Gaussian broadcast channel capacity. Let $x = x_1 + x_2$, where x_1 and x_2 are independent Gaussian signals with average powers P_1 and P_2 respectively, where $P_1 + P_2 = P$. The message intended for y_1 is transmitted through x_1, and the message intended for y_2 is transmitted through x_2. If two independent codebooks are used for x_1 and x_2, each receiver sees the other user's signal as noise. However, the transmitter knows both messages in advance. So, the channel from x_1 to y_1 can be regarded as a Gaussian channel with non-causal side information x_2, for which Costa's result applies. Thus, a transmission rate from x_1 to y_1 that is as high as if x_2 is not present can be achieved, i.e. $R_1 = I(X_1; Y_1|X_2)$. Further, the optimal x_1 is statistically independent of x_2. Thus, the channel from x_2 to y_2 still sees x_1 as independent noise, and a rate $R_2 = I(X_2; Y_2)$ is achievable. This gives an alternative derivation for the degraded Gaussian broadcast channel capacity in equations (3)-(4). Curiously, this derivation does not use the fact that y_2 is a degraded version of y_1. In fact, y_1 and y_2 may be interchanged and the following rate pair is also achievable:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{P_1}{\sigma_1^2 + P_2}\right) \tag{10}$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{P_2}{\sigma_2^2}\right). \tag{11}$$

It can be shown that, when $\sigma_1 < \sigma_2$, the above rate region is smaller than the true capacity region in equations (3)-(4).

The idea of subtracting interference at the transmitter is attractive because it is also applicable to non-degraded broadcast channels. Consider the following Gaussian vector broadcast channel:

$$\begin{aligned} \mathbf{y_1} &= H_1\mathbf{x} + \mathbf{z_1} \\ \mathbf{y_2} &= H_2\mathbf{x} + \mathbf{z_2}, \end{aligned} \tag{12}$$

where $\mathbf{x}$, $\mathbf{y_1}$ and $\mathbf{y_2}$ are vector input and outputs, H_1 and H_2 are channel matrices, and $\mathbf{z_1}$, $\mathbf{z_2}$ are Gaussian vector noises with covariance matrices $S_{z_1z_1}$ and $S_{z_2z_2}$, respectively. In general, H_1 and H_2 are not degraded versions of each other. Further, they do not necessarily have the same eigenvectors, so it is generally not possible to diagonalize H_1 and H_2 simultaneously. (An important exception is when H_1 and H_2 are ISI channels with cyclic prefix, in which case, both are Toeplitz and can be simultaneously decomposed into scalar channels by discrete Fourier transforms [19].) Nevertheless, the "dirty-paper" result can be extended to the vector case to pre-subtract multi-user interference at the transmitter, again with no increase in transmit power.

Lemma 1: Given a fixed power constraint, a Gaussian vector channel with side information $\mathbf{y} = \mathbf{x} + \mathbf{s} + \mathbf{z}$, where $\mathbf{z}$ and $\mathbf{s}$ are independent Gaussian random vectors, and $\mathbf{s}$ is known non-causally at the transmitter but not at the receiver, has the same capacity as if $\mathbf{s}$ does not exist, i.e.

$$C = \max_{p(\mathbf{u},\mathbf{x}|\mathbf{s})} \{I(\mathbf{U}; \mathbf{Y}) - I(\mathbf{U}; \mathbf{S})\} = \max_{p(x|s)} I(\mathbf{X}; \mathbf{Y}|\mathbf{S}). \tag{13}$$

Further, the capacity-achieving $\mathbf{x}$ is statistically independent of $\mathbf{s}$.

This result has been noted by several authors [20] [21] under different conditions. Lemma 1 suggests a coding scheme for the broadcast channel as shown in Figure 5. The following theorem formalizes this idea:

Theorem 1: Consider the Gaussian vector broadcast channel $\mathbf{y}_i = H_i\mathbf{x} + \mathbf{z}_i, i = 1, \cdots, K$, under a power constraint P. The following rate region is achievable:

$$\left\{ (R_1, \cdots, R_K) : R_i \le \frac{1}{2}\log\frac{\left|\sum_{k=i}^K H_i S_k H_i^T + S_{z_iz_i}\right|}{\left|\sum_{k=i+1}^K H_i S_k H_i^T + S_{z_iz_i}\right|} \right\} \tag{14}$$

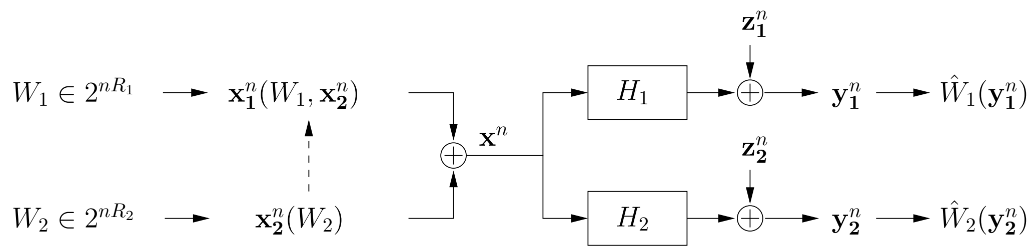


Fig. 5. Coding for vector broadcast channel

where $S_{z_iz_i}$ is the covariance matrix for $\mathbf{z}_i$, and S_i is a set of positive semi-definite matrices satisfying the constraint: $\sum_{i=1}^{K} \mathrm{tr}(S_i) \leq P$.

Proof: For simplicity, only the proof for the case $K = 2$ is presented. The extension to the general case is straightforward. Let $\mathbf{x} = \mathbf{x_1} + \mathbf{x_2}$, where $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent Gaussian vectors whose covariance matrices S_1 and S_2 satisfy $\mathrm{tr}(S_1 + S_2) \leq P$. Now, fix $\mathbf{U_2} = \mathbf{x_2}$ and choose the conditional distribution $p(\mathbf{u_1}|\mathbf{u_2}, \mathbf{x_1})$ to be such that it maximizes $I(\mathbf{U_1}; \mathbf{Y_1}) - I(\mathbf{U_1}; \mathbf{U_2})$. By Lemma 1, the maximizing distribution is such that $\mathbf{x_1}$ and $\mathbf{U_2}$ are independent. So, assuming that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent *a priori* is without loss of generality. Further, by (13), the maximizing distribution gives $I(\mathbf{U_1}; \mathbf{Y_1}) - I(\mathbf{U_1}; \mathbf{U_2}) = I(\mathbf{X_1}; \mathbf{Y_1}|\mathbf{U_2})$. Using this choice of $(\mathbf{U_1}, \mathbf{U_2})$ in Marton's region (5)-(7), the following rates are obtained: $R_1 = I(\mathbf{X_1}; \mathbf{Y_1}|\mathbf{X_2})$, $R_2 = I(\mathbf{X_2}; \mathbf{Y_2})$. The mutual information can be evaluated as:

$$R_1 = \frac{1}{2} \log \frac{|H_1 S_1 H_1^T + H_1 S_2 H_1^T + S_{z_1z_1}|}{|H_1 S_2 H_1^T + S_{z_1z_1}|} \tag{15}$$

$$R_2 = \frac{1}{2} \log \frac{|H_2 S_2 H_2^T + S_{z_2z_2}|}{|S_{z_2z_2}|}, \tag{16}$$

which is the desired result. □

This theorem is a generalization of an earlier result by Caire and Shamai [7], who essentially considered the set of rank-one S_i in their derivation of the N-by-two broadcast channel sum capacity. Theorem 1 restricts $(\mathbf{U_1}, \mathbf{U_2})$ in Marton's region to be of a special form. Although such restriction may be capacity-lossy in general, as the results in the next section show, for achieving the sum capacity of a Gaussian vector broadcast channel, this choice of $(\mathbf{U_1}, \mathbf{U_2})$ is without loss of generality. Note that finding an optimal set of S_i in (15)-(16) may not be computationally easy. Linear combinations of R_1 and R_2 are non-convex functions of (S_1, S_2). Further, the order of interference pre-subtraction is arbitrary, and it is also possible to split the transmit covariance matrix into more than two users to achieve the rate-splitting points. Caire and Shamai [7] partially circumvented the difficulty for the N-by-two broadcast channel by deriving an outer bound for the sum capacity. They assumed a particular precoding order, and by optimizing over the set of all rank-one S_i, succeeded in proving that Marton's region coincides with the outer bound for the two-user two-antenna broadcast channel. Unfortunately, their procedure does not generalize to the N-receiver case easily, and it does not reveal the structure of the optimal S_i.

In a separate effort, Ginis and Cioffi [22] demonstrated a precoding technique for an $N \times N$ broadcast channel based on a QR decomposition of the channel matrix. The QR method transforms the matrix channel into a triangular structure, and by doing so, implicitly chooses a set of S_i based on the Q matrix in the QR decomposition. This channel triangularization was also independently considered by Caire and Shamai [7], who further proved that the QR method is rate-sum optimal in both low and high SNR regions. However, this choice of S_i is sub-optimal in general.

A main goal of this paper is to find an optimal set of S_i in equations (15)-(16) that maximizes the sum capacity of a Gaussian vector broadcast channel. The key insight is that the optimal precoder has the structure of a decision-feedback equalizer.

III. Decision-feedback Precoding

A. GDFE

We begin the development by giving an information theoretical derivation of the generalized decision-feedback equalizer (GDFE). The derviation is largely tutorial in nature. It is useful in fixing the notations used in the development and for setting the stage for a subsequent generalization of GDFE. This section is based on [23].

Decision-feedback equalization (DFE) is widely used to combat intersymbol interference (ISI) in linear dispersive channels. To untangle the intersymbol interference, a decision-feedback equalizer first decodes each input symbol based on the entire received sequence, then subtract its effect on the received sequence before the decoding for the next symbol begins. Under the assumption of no error propagation and a channel non-singularity condition (that rarely occurs by accident), a generalization of decision-feedback equalizer (that often consists of several DFE's) can achieve the capacity of a Gaussian linear dispersive channel [24].

The study of the decision-feedback equalizer is related to the study of multiple access channels. If each transmitted symbol in an ISI channel is regarded as a data stream from a separate user, the decision-feedback equalizer can be thought of as a successive interference subtraction scheme for the multiple access channel. This connection can be formalized by considering a decision-feedback structure that operates on a finite block of inputs. This block-based structure, introduced in [23] as the Generalized Decision-Feedback Equalizer (GDFE), was also developed independently in [25] for the multiple access channel. This paper eventually uses the GDFE structure for the broadcast channel also.

Consider a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are Gaussian vectors. Let $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and assume that $S_{xx} > 0$. For now, assume also that the covariance matrix of $\mathbf{z}$ is non-singular. (The singular noise case is addressed in later in the paper.) In this case, it is without loss of generality to assume that $\mathbf{z} \sim \mathcal{N}(0, I)$. Shannon's noisy channel coding theorem suggests that to achieve a rate $R = I(\mathbf{X}; \mathbf{Y}) = \frac{1}{2} \log |HS_{xx}H^T + I|$, a random codebook can be constructed, in which each codeword is a sequence of Gaussian vectors generated from an i.i.d. distribution $\mathcal{N}(0, S_{xx})$. Evidently, sending a message using such a vector codebook requires joint processing of components of $\mathbf{x}$ at the encoder. Now, write $\mathbf{x}^T = [\mathbf{x_1}^T \mathbf{x_2}^T]$, and suppose further that $\mathbf{x_1}$ and $\mathbf{x_2}$ are statistically independent so that the covariance matrix S_{xx} is of the form $\begin{bmatrix} S_{x_1x_1} & 0 \\ 0 & S_{x_2x_2} \end{bmatrix}$. In this case, one might ask, is it possible to achieve a rate $R = I(\mathbf{X}; \mathbf{Y})$ using two separate codebooks with the encoding and decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$ being performed independently? The answer is yes, and the key to do so is to use a receiver based on a generalized decision-feedback equalizer.

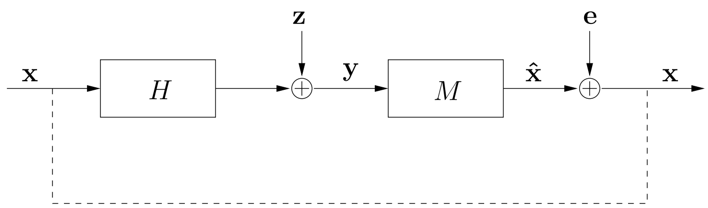


Fig. 6. MMSE estimation in a Gaussian vector channel

The development of GDFE involves three key ideas. The first idea is to recognize that in a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, the optimal decoding of $\mathbf{x}$ from $\mathbf{y}$ is related to the minimum mean-square error (MMSE) estimation of $\mathbf{x}$ given $\mathbf{y}$. Consider the setting in Figure 6, where at the output of the Gaussian vector channel, an MMSE estimator M is applied to $\mathbf{y}$ to generate $\hat{\mathbf{x}}$. First, note that the use of MMSE estimation is capacity lossless. The maximum achievable rate after MMSE estimation is $I(\mathbf{X};\hat{\mathbf{X}})$. The following argument shows that $I(\mathbf{X};\hat{\mathbf{X}}) = I(\mathbf{X};\mathbf{Y})$. The MMSE estimator for a Gaussian process is linear, so M represents a matrix multiplication. Further, let the difference between $\mathbf{x}$ and $\hat{\mathbf{x}}$ be $\mathbf{e}$. From linear estimation theory, $\mathbf{e}$ is Gaussian and is independent of $\hat{\mathbf{x}}$. So, if $I(\mathbf{X};\hat{\mathbf{X}})$ is re-written as $I(\hat{\mathbf{X}};\mathbf{X})$, it can be interpreted as an achievable rate of a Gaussian channel from $\hat{\mathbf{x}}$ to $\mathbf{x}$ with $\mathbf{e}$ as the additive noise:

$$I(\mathbf{X};\hat{\mathbf{X}}) = I(\hat{\mathbf{X}};\mathbf{X}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}, \tag{17}$$

where S_{xx} and S_{ee} are covariance matrices of $\mathbf{x}$ and $\mathbf{e}$ respectively. This mutual information is related to the capacity of the original channel. The key is the following observation [24]:

$$I(\mathbf{X};\mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{X}) = \frac{1}{2}\log\frac{|S_{yy}|}{|S_{y|x}|} = \frac{1}{2}\log\frac{|S_{yy}|}{|S_{zz}|}, \tag{18}$$

$$I(\mathbf{Y};\mathbf{X}) = H(\mathbf{X}) - H(\mathbf{X}|\mathbf{Y}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{x|y}|} = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}, \tag{19}$$

where $H(\mathbf{Y}|\mathbf{X})$ is the uncertainty in $\mathbf{y}$ given $\mathbf{x}$, so $S_{y|x} = S_{zz}$, and likewise, $H(\mathbf{X}|\mathbf{Y})$ is the uncertainty in $\mathbf{x}$ given $\mathbf{y}$, so $S_{x|y} = S_{ee}$. Since $I(\mathbf{X};\mathbf{Y}) = I(\mathbf{Y};\mathbf{X})$, this implies that

$$I(\mathbf{X};\mathbf{Y}) = I(\mathbf{Y};\mathbf{X}) = I(\mathbf{X};\hat{\mathbf{X}}) = I(\hat{\mathbf{X}};\mathbf{X}). \tag{20}$$

Now suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are independently coded with two different codebooks. The decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$, however, cannot be done on $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{2}$ separately. (Here $\hat{\mathbf{x}}^T = [\hat{\mathbf{x}}_\mathbf{1}^T \hat{\mathbf{x}}_\mathbf{2}^T]$.) To see this, write $\mathbf{e_1} = \mathbf{x_1} - \hat{\mathbf{x}}_\mathbf{1}$ and $\mathbf{e_2} = \mathbf{x_2} - \hat{\mathbf{x}}_\mathbf{2}$. Individual detections on $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{2}$ achieve $I(\mathbf{X_1};\hat{\mathbf{X}}_\mathbf{1})$ and $I(\mathbf{X_2};\hat{\mathbf{X}}_\mathbf{2})$, respectively. Because $\mathbf{e_1}$ and $\mathbf{e_2}$ are independent of $\mathbf{x_1}$ and $\mathbf{x_2}$ respectively and are both Gaussian, the argument in the previous paragraph may be repeated to conclude that individual detections on $\hat{\mathbf{x}}_\mathbf{1}$ and $\hat{\mathbf{x}}_\mathbf{2}$ achieve $\frac{1}{2}\log\left(|S_{x_1x_1}|/|S_{e_1e_1}|\right)$ and $\frac{1}{2}\log\left(|S_{x_2x_2}|/|S_{e_2e_2}|\right)$, respectively. But, $\mathbf{e_1}$ and $\mathbf{e_2}$ are not necessarily uncorrelated. So, by Hadamard's inequality, $|S_{ee}| \le |S_{e_1e_1}| \cdot |S_{e_2e_2}|$. This implies

$$\frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1e_1}|} + \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2e_2}|} \le \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|}. \tag{21}$$

Thus, although the decoding of $\mathbf{x}$ based on $\hat{\mathbf{x}}$ is capacity-lossless, the independent decoding of $\mathbf{x_1}$ based on $\hat{\mathbf{x}}_\mathbf{1}$ and decoding of $\mathbf{x_2}$ based $\hat{\mathbf{x}}_\mathbf{2}$ are capacity-lossy.

The goal of GDFE is to use a decision-feedback structure to enable the independent decoding of $\mathbf{x_1}$ and $\mathbf{x_2}$. This is accomplished by a diagonalization of the MMSE error $\mathbf{e}$, while preserving the "information" in $\hat{\mathbf{x}}$. First, let's write down the MMSE filter M,

$$
\begin{aligned}
M &= S_{xy}S_{yy}^{-1} && (22)\\
&= S_{xx}H^T(HS_{xx}H^T+I)^{-1} && (23)\\
&= (H^TH+S_{xx}^{-1})^{-1}H^T, && (24)
\end{aligned}
$$

where (22) follows from standard linear estimation theory and (24) follows from the matrix inversion lemma [26], which is used repeatedly in subsequent development:

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}. \tag{25}$$

Now, it is clear that M may be split into two parts: a matched filter H^T and an estimation filter $(H^TH+S_{xx}^{-1})^{-1}$, as shown in Figure 7. This creates a pair of channels. The forward channel goes from $\mathbf{x}$ to $\mathbf{w}$:

$$\mathbf{w} = H^TH\mathbf{x} + H^T\mathbf{z} = R_f\mathbf{x} + \mathbf{z}', \tag{26}$$

where $R_f = H^TH$. The backward channel goes from $\mathbf{w}$ to $\mathbf{x}$:

$$\mathbf{x} = (H^TH+S_{xx}^{-1})^{-1}\mathbf{w} + \mathbf{e} = R_b\mathbf{w} + \mathbf{e}, \tag{27}$$

where $R_b = (H^TH+S_{xx}^{-1})^{-1}$. The forward channel has the following property: the covariance matrix of the noise $\mathbf{z}'$ is the same as the channel matrix R_f. The second key idea in GDFE is to recognize that the backward channel has the same property as verified below:

$$
\begin{aligned}
\mathbf{E}[\mathbf{e}\mathbf{e}^T] &= \mathbf{E}[(\mathbf{x}-\hat{\mathbf{x}})(\mathbf{x}-\hat{\mathbf{x}})^T]\\
&= \mathbf{E}[(\mathbf{x}-S_{xy}S_{yy}^{-1}\mathbf{y})(\mathbf{x}-S_{xy}S_{yy}^{-1}\mathbf{y})^T]\\
&= S_{xx} - S_{xx}H^T(HS_{xx}H^T+I)^{-1}HS_{xx}\\
&= (H^TH+S_{xx}^{-1})^{-1}\\
&= R_b, && (28)
\end{aligned}
$$

where the matrix inversion lemma (25) is again used.

The goal is to diagonalize the MMSE error $\mathbf{e}$. The third key idea in GDFE is to recognize that diagonalization may be done using a block Cholesky factorization of R_b, which is simultaneously the backward channel matrix and the covariance matrix of $\mathbf{e}$:

$$R_b = G^{-1}\Delta^{-1}G^{-T}, \tag{29}$$

where $G = \begin{bmatrix} I & G_{22} \\ 0 & I \end{bmatrix}$ is a block upper triangular matrix, and $\Delta = \begin{bmatrix} \Delta_{11} & 0 \\ 0 & \Delta_{22} \end{bmatrix}$ is a block diagonal matrix. The Cholesky factorization diagonalizes $\mathbf{e}$ in the following sense. Define $\mathbf{e}' = G\mathbf{e}$:

$$\begin{bmatrix} \mathbf{e}_\mathbf{1}' \\ \mathbf{e}_\mathbf{2}' \end{bmatrix} = \begin{bmatrix} I & G_{22} \\ 0 & I \end{bmatrix} \begin{bmatrix} \mathbf{e_1} \\ \mathbf{e_2} \end{bmatrix}. \tag{30}$$

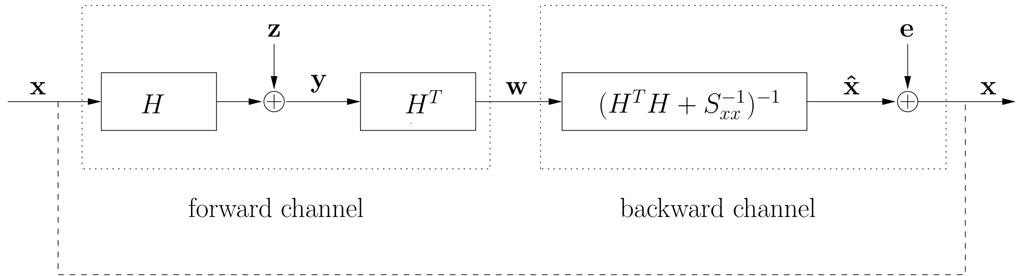


Fig. 7. Forward and backward channels

Then, the components $\mathbf{e_1'}$ and $\mathbf{e_2'}$ are uncorrelated because

$$S_{e'e'} = \mathbf{E}[\mathbf{e}'\mathbf{e}'^T] = \mathbf{E}[G\mathbf{e}(G\mathbf{e})^T] = GR_bG^T = \Delta^{-1}, \tag{31}$$

which is a block-diagonal matrix. Further, the diagonalization preserves the determinant of the covariance matrix:

$$|S_{e'e'}| = |\Delta^{-1}| = |G^{-1}\Delta^{-1}G^{-T}| = |S_{ee}|. \tag{32}$$

The next idea is to recognize that the diagonalization can be done directly by modifying the backward channel to form a decision-feedback equalizer. Because the channel matrix and the noise covariance matrix are the same, it is possible to split the channel matrix R_b into the following feedback configuration:

$$\begin{aligned} \mathbf{x} &= R_b\mathbf{w} + \mathbf{e} && (33)\\ \mathbf{x} &= G^{-1}\Delta^{-1}G^{-T}\mathbf{w} + \mathbf{e} && (34)\\ G\mathbf{x} &= \Delta^{-1}G^{-T}\mathbf{w} + G\mathbf{e} && (35)\\ \mathbf{x} &= \Delta^{-1}G^{-T}\mathbf{w} + (I-G)\mathbf{x} + \mathbf{e}'. && (36) \end{aligned}$$

Writing out the matrix computation explicitly,

$$\begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} = \begin{bmatrix} \Delta_{11}^{-1} & 0 \\ 0 & \Delta_{22}^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -G_{22}^T & I \end{bmatrix} \begin{bmatrix} \mathbf{w_1} \\ \mathbf{w_2} \end{bmatrix} + \begin{bmatrix} 0 & -G_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{e_1'} \\ \mathbf{e_2'} \end{bmatrix}. \tag{37}$$

It is now clear that the backward canonical channel is split into two independent sub-channels whose respective noises are uncorrelated. The sub-channel for $\mathbf{x_2}$ is:

$$\mathbf{x_2} = \Delta_{22}^{-1}(-G_{22}^T\mathbf{w_1} + \mathbf{w_2}) + \mathbf{e_2'} \triangleq \mathbf{x_2'} + \mathbf{e_2'}. \tag{38}$$

Once $\mathbf{x_2}$ is decoded correctly, $G_{22}\mathbf{x_2}$ can be subtracted from the sub-channel for $\mathbf{x_1}$ to form:

$$\mathbf{x_1} = \Delta_{11}^{-1}\mathbf{w_1} + \mathbf{e_1'} \triangleq \mathbf{x_1'} + \mathbf{e_1'}, \tag{39}$$

where $\mathbf{x}'$ is defined as $\mathbf{x}' \triangleq \Delta^{-1}G^{-T}\mathbf{w} + (I-G)\mathbf{x}$, and $\mathbf{x}'^T = [\mathbf{x_1'}^T \mathbf{x_2'}^T]$. This interference subtraction scheme is called a generalized decision-feedback equalizer. The GDFE structure is shown in Figure 8. The combination of $\Delta^{-1}G^{-T}$ and H^T is called the feedforward filter; $I - G$ is called the feedback filter.

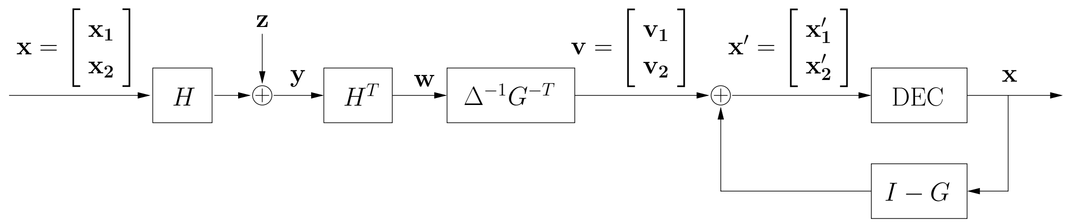


Fig. 8. Generalized decision feedback equalizer

The key result in the development of the GDFE is that the decision-feedback operation results in equivalent independent channels that have the same capacity as the original vector channel. To see this, note that the maximum achievable rate with a GDFE is $I(\mathbf{X};\mathbf{X}')$. This mutual information can be more easily computed if written as $I(\mathbf{X}';\mathbf{X})$, which can be interpreted as the capacity of the channel $\mathbf{x} = \mathbf{x}' + \mathbf{e}'$. Now, $\mathbf{e}' = G\mathbf{e}$ is independent of $\hat{\mathbf{x}}$, so it is independent of $\mathbf{w}$ and thus independent of $\mathbf{x}'$. Also, $\mathbf{e}'$ is Gaussian, so the capacity of the channel $\mathbf{x} = \mathbf{x}' + \mathbf{e}'$ is just:

$$I(\mathbf{X}';\mathbf{X}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{e'e'}|}. \tag{40}$$

This is precisely the capacity of the original channel, because by (19) and (32):

$$I(\mathbf{X};\mathbf{Y}) = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|} = \frac{1}{2}\log\frac{|S_{xx}|}{|S_{e'e'}|} = I(\mathbf{X};\mathbf{X}'). \tag{41}$$

Further, S_{xx} and $S_{e'e'}$ are both diagonal, so, $|S_{xx}| = |S_{x_1x_1}| \cdot |S_{x_2x_2}|$, and $|S_{e'e'}| = |\Delta^{-1}| = |\Delta_{11}^{-1}| \cdot |\Delta_{22}^{-1}| = |S_{e_1'e_1'}| \cdot |S_{e_2'e_2'}|$. Thus, the GDFE structure has decomposed the vector channel into two sub-channels that can be independently encoded and decoded. The capacities of the two sub-channels are:

$$\begin{aligned} R_1 &= I(\mathbf{X_1'};\mathbf{X_1}) = \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1'e_1'}|} && (42)\\ R_2 &= I(\mathbf{X_2'};\mathbf{X_2}) = \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2'e_2'}|}. && (43)\end{aligned}$$

And the sum capacity is:

$$\begin{aligned} R_1 + R_2 &= I(\mathbf{X_1'};\mathbf{X_1}) + I(\mathbf{X_2'};\mathbf{X_2}) \\ &= \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|S_{e_1'e_1'}|} + \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|S_{e_2'e_2'}|} \\ &= \frac{1}{2}\log\frac{|S_{xx}|}{|S_{ee}|} \\ &= I(\mathbf{X};\mathbf{Y}). && (44)\end{aligned}$$

Thus, GDFE is capacity lossless.

B. Precoding

For a Gaussian vector channel with independent inputs $\mathbf{x_1}$ and $\mathbf{x_2}$, the generalized decision-feedback equalizer decomposes the vector channel into two sub-channels for which encoding and decoding can be performed independently. As long as the decision-feedback operation is error-free, the sum capacity of the two sub-channels is the same as the capacity of the original vector channel. Thus, if $\mathbf{x_1}$ and $\mathbf{x_2}$ are independent, transmitter coordination is not necessary to achieve the mutual information $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y})$. On the other hand, receiver coordination is required in a decision-feedback equalizer. This is so for two reasons. First, the feedforward structure operates on the entire vector $\mathbf{y}$. Second, the feedback operation requires the correct codeword from one sub-channel to be available before the decoding of the other sub-channel. It turns out that the second problem can be averted using ideas from coding for channels with transmitter side information. In this section, a precoding scheme based on "writing-on-dirty-paper" is described. The main result is that the decision-feedback operation can be moved to the transmitter, and it is equivalent to interference "pre-subtraction".

Theorem 2: Consider a Gaussian vector channel $\mathbf{y} = \sum_{i=1}^{K} H_i \mathbf{x_i} + \mathbf{z}$, where $\mathbf{x_i}$'s are independent Gaussian vectors and $\mathbf{z} \sim \mathcal{N}(0, I)$. Under a fixed transmit covariance matrix S_{xx}, the rate $I(\mathbf{X_1}, \cdots, \mathbf{X_K}; \mathbf{Y})$ with $R_i = I(\mathbf{X_i}; \mathbf{Y} | \mathbf{X_{i+1}}, \cdots, \mathbf{X_K})$ is achievable in two ways: either using a decision-feedback structure with the knowledge of $\mathbf{x_{i+1}}, \cdots, \mathbf{x_K}$ assumed to be available before the decoding of each $\mathbf{x_i}$, or using a precoder structure with the knowledge of $\mathbf{x_{i+1}}, \cdots, \mathbf{x_K}$ assumed to be available before the encoding of each $\mathbf{x_i}$.

Proof: The development in the previous section shows that a generalized decision-feedback equalizer achieves $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y})$. To show the first part of the theorem, it is necessary to compute the individual rates of the two sub-channels. As before, let $\mathbf{x_1}$ and $\mathbf{x_2}$ be independent. Let $H = [H_1 H_2]$. (Note that in the rest of the paper, H_1 and H_2 are defined as $H^T = [H_1^T H_2^T]$. For the rest of this proof only, $H = [H_1 H_2]$.) Also, let $\mathbf{z}^T = [\mathbf{z_1}^T \mathbf{z_2}^T]$, and write the vector channel in the form of a multiple access channel:

$$\mathbf{y} = H\mathbf{x} + \mathbf{z} = [H_1 H_2] \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \end{bmatrix} + \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix}. \tag{45}$$

The block Cholesky factorization (29) may be computed explicitly:

$$(S_{xx}^{-1} + H^T H)^{-1} = \begin{bmatrix} S_{x_1x_1}^{-1} + H_1^T H_1 & H_1^T H_2 \\ H_2^T H_1 & S_{x_2x_2}^{-1} + H_2^T H_2 \end{bmatrix}^{-1} = G^{-1} \Delta^{-1} G^{-T}, \tag{46}$$

where

$$G = \begin{bmatrix} I & (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2 \\ 0 & I \end{bmatrix}, \tag{47}$$

and

$$\Delta^{-1} = \begin{bmatrix} (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} & 0 \\ 0 & (S_{x_2x_2}^{-1} + H_2^T H_2 - H_2^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2)^{-1} \end{bmatrix}. \tag{48}$$

Thus, by (32),

$$S_{e_1'e_1'} = \Delta_{11}^{-1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}. \tag{49}$$

So, from (42),

$$R_1 = I(\mathbf{X_1'};\mathbf{X_1}) = \frac{1}{2}\log\frac{|S_{x_1x_1}|}{|(S_{x_1x_1}^{-1}+H_1^TH_1)^{-1}|} = \frac{1}{2}\log|H_1S_{x_1x_1}H_1^T+I|, \tag{50}$$

where the matrix identity $|I+AB| = |I+BA|$ is used. Writing it out in another way:

$$R_1 = I(\mathbf{X_1'};\mathbf{X_1}) = I(\mathbf{X_1};\mathbf{Y}|\mathbf{X_2}). \tag{51}$$

Also,

$$\begin{aligned} S_{e_2'e_2'} &= (S_{x_2x_2}^{-1}+H_2^TH_2-H_2^TH_1(S_{x_1x_1}^{-1}+H_1^TH_1)^{-1}H_1^TH_2)^{-1}, && (52)\\ &= (S_{x_2x_2}^{-1}+H_2^T(I+H_1S_{x_1x_1}H_1^T)^{-1}H_2)^{-1}, && (53)\end{aligned}$$

where the matrix inversion lemma is used.

Thus, from (43),

$$\begin{aligned} R_2 = I(\mathbf{X_2'};\mathbf{X_2}) &= \frac{1}{2}\log\frac{|S_{x_2x_2}|}{|(S_{x_2x_2}^{-1}+H_2^T(I+H_1S_{x_1x_1}H_1^T)^{-1}H_2)^{-1}|} && (54)\\ &= \frac{1}{2}\log\frac{|H_1S_{x_1x_1}H_1^T+H_2S_{x_2x_2}H_2^T+I|}{|H_1S_{x_1x_1}H_1^T+I|}, && (55)\end{aligned}$$

which can be verified by directly multiplying out the respective terms and by repeated uses of the identity $|I+AB| = |I+BA|$. Thus,

$$R_2 = I(\mathbf{X_2'};\mathbf{X_2}) = I(\mathbf{X_2};\mathbf{Y}). \tag{56}$$

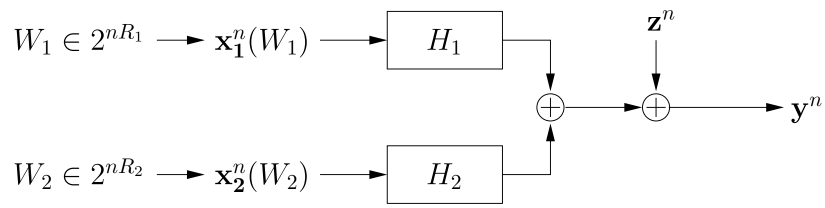
This verifies that the achievable sum rate in the multiple access channel using GDFE is

$$R_1+R_2 = I(\mathbf{X_1},\mathbf{X_2};\mathbf{Y}) = \frac{1}{2}\log|H_1S_{x_1x_1}H_1^T+H_2S_{x_2x_2}H_2^T+I|. \tag{57}$$

Therefore, the generalized decision feedback equalizer not only achieves the sum capacity of a multiple access channel, it also achieves the individual rates of a corner point in the multiple access capacity region. Interchanging the order of $\mathbf{x_1}$ and $\mathbf{x_2}$ achieves the other corner point. This, together with time-sharing or rate-splitting, allows GDFE to achieve the entire capacity region of the multiple access channel.

An induction argument generalizes the above result to more than two users. Assume that a GDFE achieves $R_i = I(\mathbf{X_i};\mathbf{Y}|\mathbf{X_{i+1}},\cdots,\mathbf{X_K})$ for a K-user multiple access channel. In a $(K+1)$-user channel, users 1 and 2 can first be considered as a super-user, and the GDFE result can be applied to the resulting K-user channel with $R_i = I(\mathbf{X_i};\mathbf{Y}|\mathbf{X_{i+1}},\cdots,\mathbf{X_{K+1}})$ for $i=3,\cdots,K$, and $R_1+R_2 = I(\mathbf{X_1},\mathbf{X_2};\mathbf{Y}|\mathbf{X_3},\cdots,\mathbf{X_{K+1}})$. Then, a separate two-user GDFE can be applied to users 1 and 2 to obtain $R_i = I(\mathbf{X_i};\mathbf{Y}|\mathbf{X_{i+1}},\cdots,\mathbf{X_{K+1}})$, for $i=1,2$.

Next, it is shown that the same rate-tuple can be achieved using a precoding structure for channels with side information at the transmitter. Consider the output of the feedforward filter, the vector $\mathbf{v}$ in Figure 8. Write $\mathbf{v}^T = [\mathbf{v_1}^T\mathbf{v_2}^T]$, and consider the capacity of the two sub-channels: one from $\mathbf{x_1}$ to



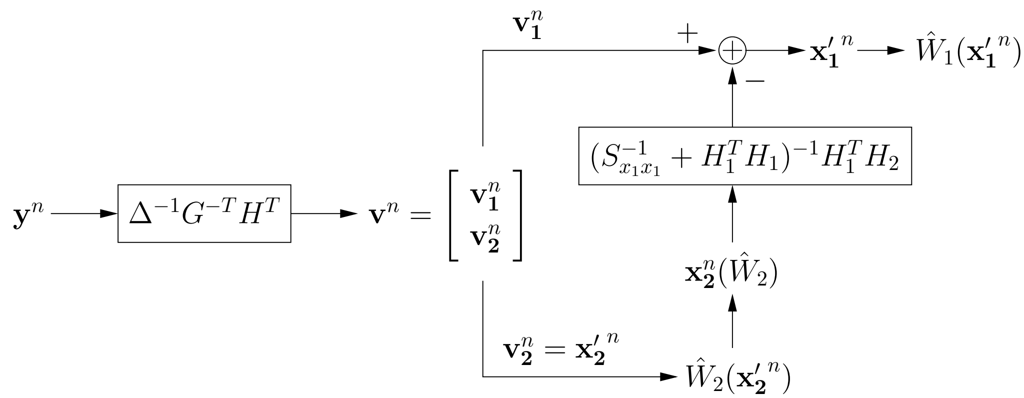


Fig. 9. Decision feedback decoding

$\mathbf{v_1}$ and the other from $\mathbf{x_2}$ to $\mathbf{v_2}$. Note that $\mathbf{v_2} = \mathbf{x_2'}$. So, the sub-channel from $\mathbf{x_2}$ to $\mathbf{v_2}$ is the same as in a GDFE:

$$R_2 = I(\mathbf{X_2}; \mathbf{V_2}) = I(\mathbf{X_2}; \mathbf{X_2'}) = I(\mathbf{X_2}; \mathbf{Y}). \tag{58}$$

Now, consider the sub-channel from $\mathbf{x_1}$ to $\mathbf{v_1}$ with $\mathbf{x_2}$ available at the transmitter. Because $\mathbf{x_2}$ is Gaussian and is independent of $\mathbf{x_1}$, Lemma 1 applies. The capacity of this sub-channel is then $R_1 = I(\mathbf{X_1}; \mathbf{V_1} | \mathbf{X_2})$. The rest of the proof shows that this conditional mutual information is equal to the corresponding data rate in GDFE: $I(\mathbf{X_1}; \mathbf{X_1'})$. Toward this end, it is necessary to explicitly compute $\mathbf{v_1}$. Since

$$\mathbf{v} = \Delta^{-1} G^{-T} H^T (H\mathbf{x} + \mathbf{z}), \tag{59}$$

using (48) and (47), $\mathbf{v_1}$ can be expressed as:

$$\mathbf{v_1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + H_2 \mathbf{x_2}) + \mathbf{z_1'}, \tag{60}$$

where $\mathbf{z}' = \Delta^{-1} G^{-T} H^T \mathbf{z}$, $\mathbf{z}'^T = [\mathbf{z_1'}^T \mathbf{z_2'}^T]$. It can be shown that $\mathbf{z_1'}$ has a covariance matrix:

$$\mathbf{E}[\mathbf{z_1'} \mathbf{z_1'}^T] = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_1 (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1}. \tag{61}$$

So, $\mathbf{v_1}$ is equivalent to

$$\mathbf{v_1} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + H_2 \mathbf{x_2} + \mathbf{z_1}), \tag{62}$$

On the other hand, $\mathbf{x_1'}$ can be computed explicitly from $\mathbf{x}' = \mathbf{v} + (I - G)\mathbf{x}$.

$$\mathbf{x_1'} = (S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T (H_1 \mathbf{x_1} + \mathbf{z_1}). \tag{63}$$

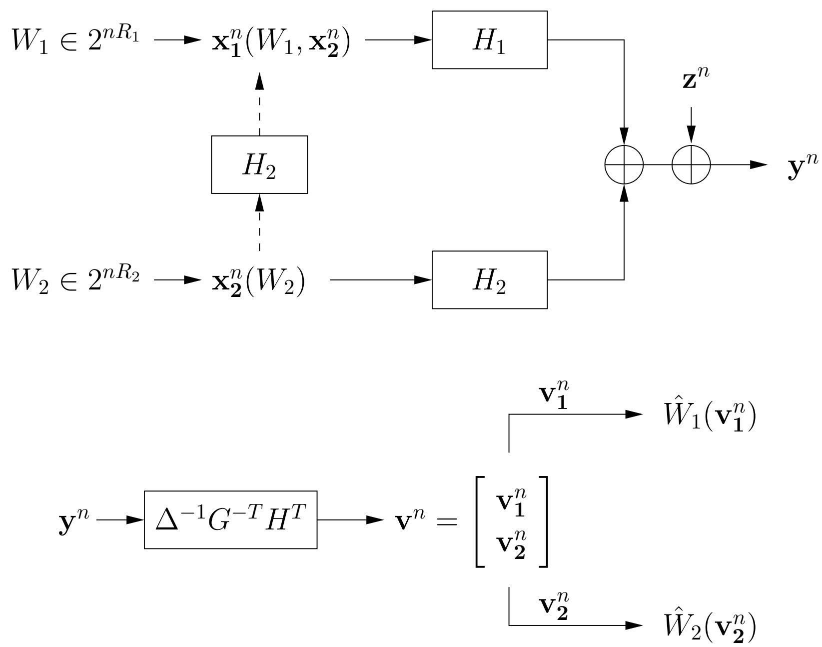


Fig. 10. Decision feedback precoding

Since $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{z_1}$ are jointly independent, it follows from (62) and (63) that

$$R_1 = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2}) = I(\mathbf{X_1}; \mathbf{X}'_\mathbf{1}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}). \tag{64}$$

Therefore, a precoder achieves the same capacity as a decision-feedback equalizer. This proof generalizes to the K-user case by a similar induction argument as before. □

Figure 9 and Figure 10 illustrate the two coding strategies for the Gaussian vector channel. Figure 9 illustrates the decision-feedback configuration. $\mathbf{x_1}$ and $\mathbf{x_2}$ are coded independently. After $\mathbf{x_2}$ is decoded, its effect, namely $(S_{x_1x_1}^{-1} + H_1^T H_1)^{-1} H_1^T H_2 \mathbf{x_2}$, is subtracted before $\mathbf{x_1}$ is decoded. This decision-feedback configuration achieves the vector channel capacity in the sense that $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}) + I(\mathbf{X_2}, \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{X}'_\mathbf{1}) + I(\mathbf{X_2}; \mathbf{X}'_\mathbf{2})$. Figure 10 illustrates the precoder configuration. In this case, $\mathbf{x_2}$ is coded as before. The channel for $\mathbf{x_1}$ is a Gaussian channel with transmitter side information $\mathbf{x_2}$, whose effect can be completely pre-subtracted. This precoder configuration achieves the vector channel capacity in the sense that $I(\mathbf{X_1}, \mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{Y}|\mathbf{X_2}) + I(\mathbf{X_2}; \mathbf{Y}) = I(\mathbf{X_1}; \mathbf{V_1}|\mathbf{X_2}) + I(\mathbf{X_2}; \mathbf{V_2})$. In the decision-feedback configuration, $\mathbf{x_2}$ is assumed to be decoded correctly before its interference is subtracted. This implies a decoding delay between the two users. Further, if an erroneous decision on $\mathbf{x_2}$ is made, error would propagate. In the precoding configuration, error propagation never occurs. However, because non-causal side information is needed, $\mathbf{x_1}$ cannot be encoded until $\mathbf{x_2}$ is available. This implies an encoding delay. The two situations are symmetric, and they are both capacity-achieving.

The decision-feedback configuration does not require transmitter coordination. So, it is naturally suited for a multiple access channel. In the precoder configuration, the feedback operation is moved to the transmitter. So, one might hope that it corresponds to a broadcast channel in which receiver coordination is not possible. This is, however, not yet true in the present setting. The capacity-

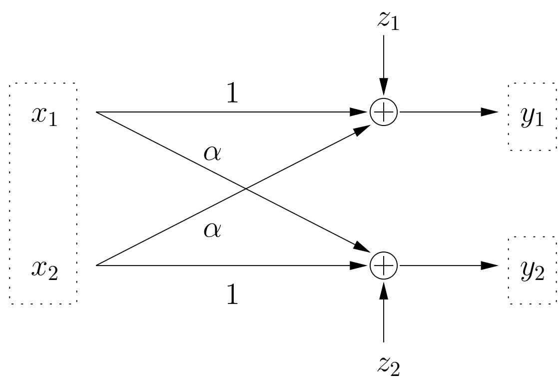


Fig. 11. A simple two-user broadcast channel

achieving precoder requires a feedforward filter that acts on the entire received vector, so receiver coordination is still needed. However, under certain conditions, the feedforward filter degenerates into a diagonal matrix, which eliminates the need for receiver coordination entirely. The condition under which this happens is the focus of the next section.

IV. Broadcast Channel Sum Capacity

A. Least Favorable Noise

The main challenge in deriving of the broadcast channel sum capacity is in finding a tight capacity outer bound. Consider the broadcast channel

$$\begin{bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{z_1} \\ \mathbf{z_2} \end{bmatrix}, \tag{65}$$

where $\mathbf{y_1}$ and $\mathbf{y_2}$ do not cooperate. Fix an input distribution $p(\mathbf{x})$. The sum capacity of the broadcast channel is clearly bounded by the capacity of the vector channel $I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2})$ where $\mathbf{y_1}$ and $\mathbf{y_2}$ cooperate. As recognized by Sato [9], this bound can be further tightened. Because $\mathbf{y_1}$ and $\mathbf{y_2}$ cannot coordinate in a broadcast channel, the broadcast channel capacity does not depend on the joint distribution $p(\mathbf{z_1}, \mathbf{z_2})$, but only on the marginals $p(\mathbf{z_1})$ and $p(\mathbf{z_2})$. This is so because two broadcast channels with the same marginals but with different joint distribution can use the same encoder and decoders and maintain the same probability of error. Therefore, the sum capacity of a broadcast channel must be bounded by the minimum mutual information:

$$R_1 + R_2 \leq \min I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2}), \tag{66}$$

where the minimization is over all $p(\mathbf{z_1}, \mathbf{z_2})$ that has the same marginal distributions as the actual noise. The minimizing noise distribution is called the "least-favorable" noise. Sato's bound is the basis for the computation of N-by-two broadcast channel capacity by Caire and Shamai [7].

The following example illustrates Sato's bound. Consider the two-user two-terminal broadcast channel shown in Figure 11, where the channel from x_1 to y_1 and the channel from x_2 to y_2 have unit gain, and the cross-over channels have a gain α. Assume that x_1 and x_2 are independent Gaussian signals, and z_1 and z_2 are Gaussian noises all with unit variance. The broadcast channel capacity is clearly bounded by $I(X_1, X_2; Y_1, Y_2)$. This mutual information is a function of the cross-over channel gain α and the correlation coefficient ρ between z_1 and z_2. Consider the case $\alpha = 0$. In this case, the least favorable noise correlation is $\rho = 0$. This is because if z_1 and z_2 were correlated, decoding of y_1 would reveal z_1 from which z_2 can be partially inferred. Such inference is possible, of course, only if y_1 and y_2 can cooperate. In a broadcast channel where y_1 and y_2 cannot take advantage of such correlation, the capacity with correlated z_1 and z_2 is the same as with uncorrelated z_1 and z_2. Thus, regardless of the actual correlation between z_1 and z_2, the broadcast channel capacity is bounded by the mutual information $I(X_1, X_2; Y_1, Y_2)$ evaluated assuming uncorrelated z_1 and z_2. Consider another case $\alpha = 1$. The least favorable noise here is the perfectly correlated noise with $\rho = 1$. This is because $\rho = 1$ implies $z_1 = z_2$ and $y_1 = y_2$. So, one of y_1 and y_2 is superfluous. If z_1 and z_2 were not perfectly correlated, (y_1, y_2) collectively would reveal more information than y_1 or y_2 alone would. Since $\rho = 1$ is the least favorable noise correlation, the broadcast channel sum capacity is bounded by the mutual information $I(X_1, X_2; Y_1, Y_2)$ assuming $\rho = 1$. This example illustrates that the least favorable noise correlation depends on the structure of the channel. The rest of this section is devoted to a characterization of the least favorable noise.

Consider the Gaussian vector channel (1): $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}, i = 1, 2$. Again, a two-user broadcast channel is considered for simplicity, and the results extend easily to the general case. Assume for now that $\mathbf{x}$ is a Gaussian vector signal with a fixed covariance matrix S_{xx}, and $\mathbf{z_1}$ $\mathbf{z_2}$ are jointly Gaussian noises with a marginal distribution $\mathbf{z_i} \sim \mathcal{N}(0, I)$. Then, the task of finding the least favorable noise correlation can be formulated as an optimization problem. Let $H^T = [H_1^T H_2^T]$. The optimization problem is:

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} \quad & S_{zz}^{(i)} = I, \qquad i = 1, 2, \\ & S_{zz} \geq 0, \end{aligned} \tag{67}$$

where S_{zz} is the covariance matrix for $\mathbf{z}$ with $\mathbf{z}^T = [\mathbf{z_1}^T \mathbf{z_2}^T]$, and $S_{zz}^{(i)}$ refers to the ith block-diagonal term of S_{zz}. The optimization is over all off-diagonal terms of S_{zz} subject to the constraint that S_{zz} is positive semi-definite.

In writing down the optimization problem (67), it is tacitly assumed that the minimizing S_{zz} is strictly positive definite, so that $|S_{zz}| > 0$. This is an additional assumption that is made at first, but will eventually be removed. Note that the minimizing S_{zz} is often singular. For example, for the two-user broadcast channel considered earlier with $\alpha = 1$, the least favorable noise has a covariance matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which is singular. A sufficient condition for the minimizing S_{zz} to be non-singular is that $|HS_{xx}H^T| > 0$. This is because whenever $|S_{zz}| = 0$, it must also be true that $|HS_{xx}H^T + S_{zz}| = 0$, (as otherwise the mutual information goes to infinity.) But $|HS_{xx}H^T + S_{zz}|$ cannot be zero unless $|HS_{xx}H^T|$ is zero. Thus, $|HS_{xx}H^T| > 0$ is sufficient to ensure that $|S_{zz}| > 0$. This sufficient condition holds, for example, when both H and S_{xx} are full rank.

The following lemma characterizes an optimality condition for the least favorable noise assuming that such a noise is non-singular. For now, the transmit signal for the broadcast channel $\mathbf{x}$ is assumed to be Gaussian with a fixed covariance matrix. It will be shown later that the Gaussian restriction is without loss of generality.

Lemma 2: Consider a Gaussian vector broadcast channel $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}, i = 1, 2$, where $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$ and $\mathbf{z_i} \sim \mathcal{N}(0, I)$. Let $H^T = [H_1^T H_2^T]$. Then, the least favorable noise distribution that minimizes $I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2})$ is jointly Gaussian. Further, if the minimizing S_{zz} is non-singular, then, the least favorable noise has a covariance matrix S_{zz} such that $S_{zz}^{-1} - (HS_{xx}H^T + S_{zz})^{-1}$ is a block-diagonal matrix. Conversely, any Gaussian noise with a covariance matrix S_{zz} that satisfies the diagonalization condition and has $S_{zz}^{(i)} = I$ is a least favorable noise.

Proof: Fix a Gaussian input distribution $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and fix a noise covariance matrix S_{zz}. Let $\mathbf{z} \sim \mathcal{N}(0, S_{zz})$ be a Gaussian random vector, and let $\mathbf{z}'$ be any other random vector with the same covariance matrix, but with possibly a different distribution. Then, $I(\mathbf{X}; H\mathbf{X}+\mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X}+\mathbf{Z}')$. This fact is proved in [27] and [28]. Thus, to minimize $I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2})$, it is without loss of generality to restrict attention to $\mathbf{z_1}, \cdots, \mathbf{z_2}$ that are jointly Gaussian. In this case, the cooperative capacity is just $\frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$. So, the least favorable noise is the solution to the optimization problem (67).

The objective function in the optimization problem is convex in the set of semi-definite matrices S_{zz}. The constraints are convex in S_{zz}, and they satisfy the constrained quantification condition. Thus, the Karush-Kuhn-Tucker (KKT) condition is a necessary and sufficient condition for optimality. To derive the KKT condition, form the Lagrangian:

$$L(S_{zz}, \Psi_1, \Psi_2, \Phi) = \log |HS_{xx}H^T + S_{zz}| - \log |S_{zz}| + \sum_{i=1}^{2} \mathrm{tr}(\Psi_i(S_{zz}^{(i)} - I)) - \mathrm{tr}(\Phi S_{zz}), \tag{68}$$

where $(\Psi_1, \cdots, \Psi_K)$ are dual variables associated with the block-diagonal constraints, and Φ is a dual variable associated with the semi-definite constraint. (Ψ_1, Ψ_2, Φ are positive semi-definite matrices.) The coefficient $\frac{1}{2}$ is omitted for simplicity. Setting $\partial L/\partial S_{zz}$ to zero:

$$0 = \frac{\partial L}{\partial S_{zz}} = (HS_{xx}H^T + S_{zz})^{-1} - S_{zz}^{-1} + \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix} - \Phi. \tag{69}$$

The minimizing S_{zz} is assumed to be positive definite. So, by the complementary slackness condition (see Appendix A), $\Phi = 0$. Thus, at the optimum, the following block-diagonal condition must be satisfied:

$$S_{zz}^{-1} - (HS_{xx}H^T + S_{zz})^{-1} = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}. \tag{70}$$

Conversely, this block-diagonal condition combined with the constraints in the original problem form the KKT condition, which is sufficient for optimality. Thus, if a noise covariance matrix satisfies (70), it must be a least favorable noise. □

Note that the diagonalization condition may be written in a different form. If assuming, in addition, that $HS_{xx}H^T$ is non-singular and Ψ_1, Ψ_2 are invertible, (70) may be re-written using the matrix inversion lemma as follows:

$$S_{zz} + S_{zz}(HS_{xx}H^T)^{-1}S_{zz} = \begin{bmatrix} \Psi_1^{-1} & 0 \\ 0 & \Psi_2^{-1} \end{bmatrix}. \tag{71}$$

Curiously, this equation resembles a Ricatti equation. Neither (70) nor (71) appears to have a closed-form solution.

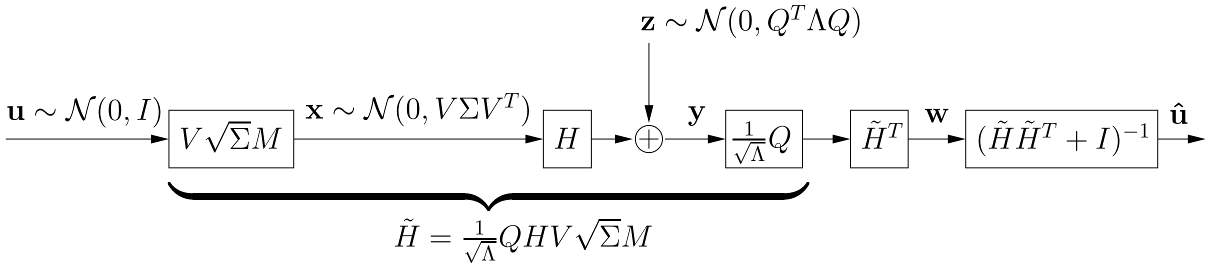


Fig. 12. GDFE with transmit filter

B. GDFE with Non-Singular Least Favorable Noise

The main result of this paper is that the cooperative capacity of the Gaussian vector channel with a least favorable noise is achievable for the Gaussian broadcast channel. Toward this end, it is shown that a generalized decision feedback precoder designed for the least favorable noise does not require receiver coordination in the sense that not only can the feedback operation be moved to the transmitter by precoding, but the feedforward matrix can also be made to have a block-diagonal structure that totally eliminates the need for receiver coordination. The derivation is most transparent when the least-favorable noise is non-singular. In this case, the least-favorable noise satisfies a noise diagonalization condition (70). The non-singular assumption is removed in the next section.

Consider a generalized decision-feedback equalizer designed for the Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$. For now, assume that $\mathbf{x}$ is Gaussian. Also assume that the covariance matrix of $\mathbf{z}$ is full rank. If the noise covariance matrix S_{zz} is not block-diagonal, an implementation of the GDFE requires noise whitening as a first step. Suppose that the noise covariance matrix has an eigenvalue decomposition:

$$S_{zz} = Q^T \Lambda Q, \tag{72}$$

where Q is an orthogonal matrix and Λ is a diagonal matrix, then $\frac{1}{\sqrt{\Lambda}}Q$ is the appropriate noise whitening filter. If in addition, the transmitter covariance matrix S_{xx} is also not block-diagonal, then a Gaussian source $\mathbf{u}$ and a transmit filter B can be created such that $S_{uu} = I$ and $\mathbf{x} = B\mathbf{u}$. Let

$$S_{xx} = V\Sigma V^T \tag{73}$$

be an eigenvalue decomposition of the transmit covariance matrix S_{xx}. The appropriate transmit filter must be of the form:

$$B = V\sqrt{\Sigma}M \tag{74}$$

where M is an arbitrary orthogonal matrix, so that $S_{xx} = BS_{uu}B^T = V\Sigma V^T$. A different generalized decision-feedback equalizer can be designed for each choice of M.

Lemma 3: Consider the Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, where $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$, and H is a square matrix. Fix a Gaussian source $\mathbf{u} \sim \mathcal{N}(0, I)$. If the noise covariance matrix S_{zz} is non-singular, and it satisfies the condition that $S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1}$ is a block-diagonal matrix, then there exists a transmit filter B such that $\mathbf{x} = B\mathbf{u}$ has a covariance matrix S_{xx} and the induced generalized decision-feedback equalizer has feedforward filter that is a block-diagonal.

Proof: The GDFE configuration is as shown in Figure 12. Let $S_{xx} = V\Sigma V^T$ and $S_{zz} = Q^T\Lambda Q$. As stated before, the transmit filter must be of the form $B = V\sqrt{\Sigma}M$, where M is an orthogonal matrix. The noise whitening filter is $\frac{1}{\sqrt{\Lambda}}Q$. The combined transmit filter and the noise whitening filter give the following effective channel:

$$\widetilde{H} = \frac{1}{\sqrt{\Lambda}}QHV\sqrt{\Sigma}M. \tag{75}$$

The GDFE depends on the following Cholesky factorization:

$$\begin{aligned} G^{-1}\Delta^{-1}G^{-T} &= (\widetilde{H}^T\widetilde{H} + I)^{-1} & (76)\\ &= \left(M^T\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma}M + I\right)^{-1} & (77)\\ &= M^T\left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma} + I\right)^{-1}M. & (78)\end{aligned}$$

Now, choose a square matrix R such that

$$R^TR = \left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma} + I\right)^{-1}. \tag{79}$$

(For example, R can be chosen to be a triangular matrix using a Cholesky factorization.) Now, because the right-hand side of the above is positive definite, all square matrices C that satisfy $C^TC = \left(\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}QHV\sqrt{\Sigma} + I\right)^{-1}$ must be of the form $C = UR$ where U is an orthogonal matrix [29]. Therefore, the Cholesky factorization (78) can be written as:

$$G^{-1}\Delta^{-1}G^{-T} = M^TR^TU^TURM, \tag{80}$$

where URM is a block-lower-triangular matrix. For a fixed M, it is possible to choose a U to make URM block-triangular. Such a U can be found via a block QR-factorization of RM. Similarly, for each fixed U, it is possible to choose a M that makes URM block-triangular. Such a M can be found by a block QR-factorization of $(UR)^T$.

The feedforward filter of a GDFE, denoted as F, can now be computed as follows:

$$\begin{aligned} F &= \Delta^{-1}G^{-T}\widetilde{H}^T\frac{1}{\sqrt{\Lambda}}Q & (81)\\ &= \Delta^{-\frac{1}{2}}URMM^T\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q & (82)\\ &= \Delta^{-\frac{1}{2}}UR\sqrt{\Sigma}V^TH^TQ^T\Lambda^{-1}Q. & (83)\end{aligned}$$

Next, it is shown that the condition under which there exists a suitable U to make the feedforward filter F a block-diagonal matrix is the same as the diagonalization condition on the noise covariance matrix. First, assume that $S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1}$ is block-diagonal. Then,

$$\begin{aligned} \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix} &= S_{zz}^{-1} - (S_{zz} + HS_{xx}H^T)^{-1} & (84)\\ &= Q^T\Lambda^{-1}Q - (Q^T\Lambda Q + HV\Sigma V^TH^T)^{-1} \end{aligned}$$

$$= Q^T \Lambda^{-\frac{1}{2}} \left(I - \left(I + \Lambda^{-\frac{1}{2}} Q H V \Sigma V^T H^T Q^T \Lambda^{-\frac{1}{2}} \right)^{-1} \right) \Lambda^{-\frac{1}{2}} Q$$
$$= Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} \left(I + \sqrt{\Sigma} V^T H^T Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} \right)^{-1}$$
$$\sqrt{\Sigma} V^T H^T Q^T \Lambda^{-1} Q$$

where the matrix inversion lemma is used in the last step. Now, substituting (79) into the above:

$$Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} R^T R \sqrt{\Sigma} V^T H^T Q^T \Lambda^{-1} Q = \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix}. \tag{85}$$

Now, without loss of generality, H can be assumed to be a square matrix. Assuming that H is a square matrix, $R\sqrt{\Sigma}V^T H^T Q^T$ is also square. So, it must be of the form $U'D$, where U' is orthogonal and $D = \text{diag}\{\sqrt{\Psi}_1, \sqrt{\Psi}_2\}$:

$$R\sqrt{\Sigma} V^T H^T Q^T \Lambda^{-1} Q = U' \begin{bmatrix} \sqrt{\Psi}_1 & 0 \\ 0 & \sqrt{\Psi}_2 \end{bmatrix}. \tag{86}$$

But, this is exactly the diagonalization condition for F. By choosing $U = U'^T$ in (83), F becomes:

$$F = \Delta^{-\frac{1}{2}} U'^T R \sqrt{\Sigma} V^T H^T Q^T \Lambda^{-1} Q \tag{87}$$
$$= \Delta^{-\frac{1}{2}} \begin{bmatrix} \sqrt{\Psi}_1 & & 0 \\ & \ddots & \\ 0 & & \sqrt{\Psi}_K \end{bmatrix}. \tag{88}$$

which is block-diagonal. Finally, an appropriate transmit filter B can be found by finding an M that makes URM block lower-triangular. This is possible by performing the following QR-factorization: $R^T U^T = MT$, where T is upper-triangular and M is orthogonal. Then, $URM = T^T$ is lower-triangular. □

Combining Lemma 2 and Lemma 3, it is now clear that the Gaussian vector channel with the least favorable noise admits a GDFE structure whose feedforward filter is block-diagonal, provided that the least favorable noise is non-singular. This means that at the feedforward stage, only individual processing of $\mathbf{y_i}$ is needed. This, together with the fact that decision-feedback can be moved to the transmitter as a precoder, completely eliminates the need for receiver cooperation. To complete the argument, it is shown next that this is true even when the least-favorable noise is singular.

C. GDFE with Singular Noise

The goal of this section is to prove the following:

Lemma 4: Consider the Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$, where $\mathbf{x} \sim \mathcal{N}(0, S_{xx})$ and H is a square matrix. There exists a GDFE structure for the Gaussian vector channel with a block-diagonal feedforward matrix, if and only if S_{zz} is the minimizing solution to (67).

Proof: Let S_{zz} be the solution to the minimization problem (67)

$$\min_{S_{zz}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}. \tag{89}$$

The result follows directly from Lemma 2 and Lemma 3 when the minimizing S_{zz} is singular. To show that there exists a GDFE structure whose feedforward section is diagonal even when S_{zz} is singular, both the KKT condition and the GDFE structure need to be generalized.

The decision-feedback equalizer that is required to accommodate singular noise differs from the conventional structure in one crucial aspect. A conventional DFE never processes more output dimensions than input dimensions. For example, in a channel with two transmit antennas and three receive antennas, a conventional DFE always reduces the two-by-three channel to a two-by-two channel by receiver joint processing. However, in a broadcast channel, receivers cannot cooperate and joint processing is not possible. Thus, a non-trivial generalization of the conventional DFE structure is required.

Without loss of generality, S_{xx} is assumed to be fixed and full-rank. Let $S_{xx} = V\Sigma V^T$ be an eigenvalue decomposition. It is convenient to regard $HV\sqrt{\Sigma}M$ as the effective channel, and the input covariance matrix as an identity matrix. The choice of M will be made later.

Suppose that S_{zz} is a low-rank solution to the minimization problem (89). Decompose S_{zz} in the following form:

$$S_{zz} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_{\tilde{z}\tilde{z}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix}, \tag{90}$$

where $S_{\tilde{z}\tilde{z}}$ is invertible and $[\ U_1\ U_2\ ]$ is an orthonormal matrix. The null space of S_{zz} must be a subspace of the null space of the channel, (because otherwise the capacity would be infinity). Thus, it must be possible to express the effective matrix as:

$$HV\sqrt{\Sigma}M = U_1 H'. \tag{91}$$

The minimization problem now becomes

$$\begin{aligned} \min_{S_{\tilde{z}\tilde{z}}} \quad & \frac{1}{2}\log\frac{|H'H'^T + S_{\tilde{z}\tilde{z}}|}{|S_{\tilde{z}\tilde{z}}|} \\ \text{s.t.} \quad & U_1 S_{\tilde{z}\tilde{z}} U_1^T \text{ is diagonal.} \end{aligned} \tag{92}$$

A necessary condition for the least-favorable noise is

$$S_{\tilde{z}\tilde{z}}^{-1} - (H'H'^T + S_{\tilde{z}\tilde{z}})^{-1} = U_1^T \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix} U_1. \tag{93}$$

The objective is to show that if the noise covariance matrix satisfies the above condition, then there exists a decision-feedback equalizer whose feedforward matrix is diagonal.

Recall that the derivation of the decision-feedback equalizer is based on minimum mean-square error (MMSE) estimation. The key observation is that MMSE estimator is not unique when both the channel and the noise are rank deficient. Consider the MMSE estimation of $\mathbf{x}$ given $\mathbf{y} = H\mathbf{x} + \mathbf{z}$. The MMSE estimation is a matrix multiplication $\hat{\mathbf{x}} = W\mathbf{y}$, where W satisfies a normal equation:

$$WS_{yy} = S_{xy}. \tag{94}$$

Because both S_{zz} and H are low rank, S_{yy} is also low rank. It is not difficult to verify that

$$W = H'^T(H'H'^T + S_{\tilde{z}\tilde{z}})^{-1}U_1^T + SU_2^T \tag{95}$$

satisfies the normal equation for any choice of S. A different DFE can be designed for each choice of S. Thus, to prove the Lemma, it is only necessary to show that there exists one choice of S (along with a choice of M) that makes the DFE feedforward matrix diagonal.

The first step in the design of a decision-feedback equalizer is noise whitening. Define

$$\hat{H} = S_{\tilde{z}\tilde{z}}^{-\frac{1}{2}} H'. \tag{96}$$

The MMSE estimator matrix W can be re-written as

$$W = S_{xx}\hat{H}^T(\hat{H}\hat{H}^T + I)^{-1}S_{\tilde{z}\tilde{z}}^{-\frac{1}{2}}U_1^T + SU_2^T \tag{97}$$

The structure of the feedforward matrix involves a Cholesky factorization of $R_b = \mathbf{E}[(\mathbf{X}-\hat{\mathbf{X}})(\mathbf{X}-\hat{\mathbf{X}})^T]$. It turns out that R_b is independent of the choice of S. Using the matrix inversion lemma, it can be shown that

$$R_b = (\hat{H}^T\hat{H} + I)^{-1}. \tag{98}$$

Further, W can be re-written as follows:

$$W = (\hat{H}^T\hat{H} + I)^{-1}\hat{H}^T S_{\tilde{z}\tilde{z}}^{-\frac{1}{2}}U_1^T + SU_2^T. \tag{99}$$

In a decision-feedback equalizer, half of the Cholesky factorization $R_b = G^{-1}\Delta^{-1}G^{-T}$ is placed in the feedback section, and the remaining half is placed in the feedforward section. Thus, the feedforward matrix has the form:

$$F = \Delta^{-1}G^{-T}\hat{H}^T S_{\tilde{z}\tilde{z}}^{-\frac{1}{2}}U_1^T + S'U_2^T. \tag{100}$$

The goal is to use the generalized least-favorable noise condition (93) to show that F can be made diagonal. Using the matrix inversion lemma,

$$\begin{aligned} U_1^T \begin{bmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{bmatrix} U_1 &= S_{\tilde{z}\tilde{z}}^{-1} - (\tilde{H}\tilde{H}^T + S_{\tilde{z}\tilde{z}})^{-1} \\ &= \hat{H}(I + \hat{H}\hat{H}^T)^{-1}\hat{H}^T. \end{aligned} \tag{101}$$

Following the same derivation as in (83) - (88), it is not difficult to show that by an appropriate choice of M, the Cholesky factorization of R_b can be made to give:

$$\Delta^{-\frac{1}{2}}G^{-T}\hat{H}^T S_{\tilde{z}\tilde{z}}^{-\frac{1}{2}} = \begin{bmatrix} \sqrt{\Psi_1} & 0 \\ 0 & \sqrt{\Psi_2} \end{bmatrix}. \tag{102}$$

Therefore, by choosing

$$S' = \Delta^{-\frac{1}{2}} \begin{bmatrix} \sqrt{\Psi_1} & 0 \\ 0 & \sqrt{\Psi_2} \end{bmatrix} U_2, \tag{103}$$

the feedforward matrix becomes

$$F = \Delta^{-\frac{1}{2}} \begin{bmatrix} \sqrt{\Psi_1} & 0 \\ 0 & \sqrt{\Psi_2} \end{bmatrix} (U_1U_1^T + U_2U_2^T), \tag{104}$$

which is diagonal since

$$U_1U_1^T + U_2U_2^T = [\ U_1\ U_1\ ] \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} = I. \tag{105}$$

□

To summarize, when the least-favorable noise is singular, it must satisfy a modified KKT condition (67). The decision-feedback equalizer structure with the singular noise is not unique. However, among the class of decision-feedback equalizers, there exists one whose feedforward matrix is diagonal. Thus, in a Gaussian vector broadcast channel, the minimum mutual information $I(\mathbf{X}; \mathbf{Y_1}, \mathbf{Y_2})$ is an achievable sum rate, even when the least favorable noise covariance matrix is singular. This rate is achieved under a fixed input covariance S_{xx}. So, one might expect the sum capacity of the broadcast channel to be the above mutual information maximized over all S_{xx} subject to a power constraint. This is we prove next.

D. Sum Capacity

The development so far contains the simplifying assumption that the input distribution is Gaussian. To see that the restriction is without loss of generality, a result concerning the saddle-point is useful. Consider the mutual information expression $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$, where $\mathbf{X}$ and $\mathbf{Z}$ are independent. Let $\mathcal{K}_x$ and $\mathcal{K}_z$ be constraint sets for $\mathbf{X}$ and $\mathbf{Z}$. If some $(p(\mathbf{x}), p(\mathbf{z}))$ is such that for all $p(\mathbf{x}') \in \mathcal{K}_x$ and $p(\mathbf{z}') \in \mathcal{K}_z$,

$$I(\mathbf{X}'; H\mathbf{X}' + \mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}') \tag{106}$$

then $(p(\mathbf{x}), p(\mathbf{z}))$ is called a saddle-point. The main result concerning the saddle-point is the following:

Lemma 5 ([28]) The mutual information expression $I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$, where $p(\mathbf{x}) \in \mathcal{K}_x$ and $p(\mathbf{z}) \in \mathcal{K}_z$ are convex constraints, has at least one saddle-point. Further, there exists a saddle-point whose distributions are Gaussian.

The proof of this result can be found in [28]. It goes as follows: First, it is shown that the search for the saddle-point can be restricted to Gaussian distributions without loss of generality. With Gaussian distributions, the mutual information can be written as $\frac{1}{2}\log|HS_{xx}H^T + S_{zz}|/|S_{zz}|$. Because $\log|\cdot|$ is a concave function over the set of positive definite matrices, $\frac{1}{2}\log|HS_{xx}H^T + S_{zz}|/|S_{zz}|$ is convex in S_{zz} and concave in S_{xx}. The constraints are convex. So, from a minimax theorem in game theory [30], there exists a saddle-point (S_{xx}, S_{zz}) such that

$$\frac{1}{2}\log\frac{|HS'_{xx}H^T + S_{zz}|}{|S_{zz}|} \leq \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \leq \frac{1}{2}\log\frac{|HS_{xx}H^T + S'_{zz}|}{|S'_{zz}|}, \tag{107}$$

for all (S'_{xx}, S'_{zz}) in the constraint sets.

A saddle-point (when exists) is the solution to the following max-min problem:

$$\max_{p(\mathbf{x})}\min_{p(\mathbf{z})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}). \tag{108}$$

This can be easily seen as follows. Suppose $(\mathbf{X}, \mathbf{Z})$ is a saddle-point. Then, $\min_{p(\mathbf{z}'')} I(\mathbf{X}'; H\mathbf{X}'+\mathbf{Z}'') \leq I(\mathbf{X}'; H\mathbf{X}'+\mathbf{Z}) \leq I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$. So $\max_{p(\mathbf{x}')}\min_{p(\mathbf{z}')} I(\mathbf{X}'; H\mathbf{X}'+\mathbf{Z}') \leq I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$. On the other hand, fixing $p(\mathbf{x})$ gives $\min_{p(\mathbf{z})} I(\mathbf{X}; H\mathbf{X}+\mathbf{Z}') = I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$. So, $\max_{p(\mathbf{x}')} \min_{p(\mathbf{z}')} I(\mathbf{X}'; H\mathbf{X}'+\mathbf{Z}') = I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$. By the same argument, the saddle-point is also the solution to the min-max problem:

$$\min_{p(\mathbf{z})} \max_{p(\mathbf{x})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}). \tag{109}$$

For any arbitrary function $f(x, y)$, it is always true that $\min_x \max_y f(x, y) \geq \max_y \min_x f(x, y)$. However, if a saddle-point exists, then max-min equals min-max:

$$\max_{S_{xx}} \min_{S_{zz}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} = \min_{S_{zz}} \max_{S_{xx}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}. \tag{110}$$

The main result of this paper is that max-min corresponds to achievability, min-max corresponds to the converse, and the saddle-point corresponds to the sum capacity of a Gaussian vector broadcast channel.

Theorem 3: Consider a Gaussian vector broadcast channel $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}$, $i = 1, \cdots, K$, under a power constraint P. Let $H^T = [H_1^T \cdots H_K^T]$. The sum capacity is a saddle-point of the mutual information $\frac{1}{2} \log |HS_{xx}H^T + S_{zz}|/|S_{zz}|$ with the following constraints: S_{zz} has block-diagonal entries that are the covariance matrices of $\mathbf{z_1}, \cdots, \mathbf{z_K}$, and S_{xx} satisfies $\mathrm{tr}(S_{xx}) \leq P$.

Proof: First, the converse: Sato's outer bound states that the broadcast channel sum capacity is bounded by the capacity of any discrete memoryless channel whose noise marginal distributions are equal to $p(\mathbf{z_i})$. The tightest outer bound is then the capacity of the channel with the least favorable noise correlation. The capacity of a discrete memoryless channel is $\max_{p(\mathbf{x})} I(\mathbf{X}; \mathbf{Y_1}, \cdots, \mathbf{Y_K})$, so:

$$C \leq \min_{p(\mathbf{z})} \max_{p(\mathbf{x})} I(\mathbf{X}; H\mathbf{X} + \mathbf{Z}), \tag{111}$$

where the maximization is over the power constraint $\mathbf{E}[\mathbf{X}^T\mathbf{X}] \leq P$, and the minimization is over all noise distributions whose marginals are the same as the actual noise. The solution to this minimax problem is the saddle-point for $I(\mathbf{X}; H\mathbf{X} + \mathbf{Z})$. Since the constraint sets are convex, by Lemma 5, a saddle-point exists. Further, the saddle-point can be chosen to be Gaussian, so the outer bound can be written as

$$C \leq \min_{S_{zz}} \max_{S_{xx}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}, \tag{112}$$

where S_{xx} belongs to the set of positive semi-definite matrices satisfying the power constraint $\mathrm{tr}(S_{xx}) \leq P$, and S_{zz} belongs to the set of noise covariance matrices with $S_{zz}^{(i)} = \mathbf{E}[\mathbf{z_i}\mathbf{z_i}^T]$, $i = 1, \cdots, K$, as block-diagonal terms.

Next, the achievability: the existence of a saddle-point implies that min-max equals max-min. So, it is only necessary to show that

$$C \geq \max_{S_{xx}} \min_{S_{zz}} \frac{1}{2} \log \frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|}. \tag{113}$$

Since the saddle-point can be chosen to be Gaussian, the development leading to the theorem, which restricts consideration to Gaussian inputs, is without loss of generality. Further, Lemma 3 and Lemma 4 requires the channel matrix to be square. If there are more receive antennas than transmit antennas, zeros can be padded to H without affecting capacity, so H can be made square. If there are more transmit antennas than receive antennas, because S_{xx} is a water-filling covariance matrix with respect to H, the rank of S_{xx} is bounded by the number of receive antennas. Then, the null space of S_{xx} may be deleted, and H can be made equivalent to a square matrix. In either case, the condition of square H can be satisfied.

Now, at the saddle-point, S_{zz} is a least favorable noise for S_{xx}. So, by Lemma 2, Lemma 3 and Lemma 4, there must exist an appropriate transmit filter B such that a GDFE designed for this B and S_{zz} has a block-diagonal feedforward matrix. Consider now the precoding configuration of the GDFE. The feedforward section is block-diagonal. The feedback section is moved to the transmitter. So, the decoding operations of $\mathbf{y_1}, \cdots, \mathbf{y_K}$ are completely independent of each other. Further, because the feedback filter is block-diagonal, the GDFE receiver is oblivious of the correlation between $\mathbf{z_i}$'s. Thus, although the actual noise distribution may not have the same joint distribution as the least favorable noise, because the marginal distributions are the same, a GDFE precoder designed for the least favorable noise performs as well as with the actual noise. Since by Theorem 2, this GDFE precoder achieves $I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$, so $\min_{S_{zz}} I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$ is achievable. Further, it is possible to maximize the above over S_{xx}. Therefore, the outer bound (113) is achievable. □

Note that the GDFE transmit filter B designed for the least favorable noise also identifies the set of sum capacity-achieving S_i in Theorem 1. Let $B = [B_1 \cdots B_K]$. Set $S_1 = B_1 B_1^T$, $\cdots$, $S_K = B_K B_K^T$. Then, it is easy to verify that the sum capacity is achieved with $R_i = \frac{1}{2}\log|\sum_{k=i}^{K} H_i S_k H_i^T + I|/|\sum_{k=i+1}^{K} H_i S_k H_i^T + I|$.

Theorem 3 suggests the following game-theoretical interpretation of the Gaussian vector broadcast channel. There are two players in the game. A signal player chooses a S_{xx} to maximize $I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$ subject to the constraint $\mathrm{tr}(S_{xx}) \le P$. A noise player chooses a fictitious noise correlation in S_{zz} to minimize $I(\mathbf{X}; H\mathbf{X}+\mathbf{Z})$ subject to the constraint $S_{zz}^{(i)} = I$. A Nash equilibrium in the game is a set of strategies such that each player's strategy is the best response to the other player's strategy. The Nash equilibrium in this mutual information game exists, and the Nash equilibrium is the sum capacity of the Gaussian vector broadcast channel.

The saddle-point property of the Gaussian broadcast channel sum capacity implies that the capacity achieving (S_{xx}, S_{zz}) is such that S_{xx} is the water-filling covariance matrix for S_{zz}, and S_{zz} is the least favorable noise covariance matrix for S_{xx}. In fact, the converse is also true. If a set of (S_{xx}, S_{zz}) can be found such that S_{xx} is the water-filling covariance for S_{zz}, and S_{zz} is the least favorable noise for S_{xx}, then (S_{xx}, S_{zz}) constitutes a saddle-point. This is because the mutual information is a concave-convex function, and the two KKT conditions, corresponding to the two optimization problems are, collectively, sufficient and necessary at the saddle-point [31] [32]. Thus, the computation of the saddle-point is equivalent to simultaneously solving the water-filling problem and the least favorable noise problem.

One might suspect that the following algorithm can be used to find a saddle-point numerically. The idea is to iteratively compute the best input covariance matrix S_{xx} for a given noise covariance, then compute the least favorable noise covariance matrix S_{zz} for the given input covariance. If the iterative process converges, both KKT conditions are satisfied, and the limit must be a saddle-point of $\frac{1}{2}\log|HS_{xx}H^T + S_{zz}|/|S_{zz}|$. Although such an iterative min-max procedure is not guaranteed to converge for a general game even when the pay-off function is concave-convex, the iterative procedure appears to work well in practice for this particular problem. The convex-concave nature of the problem also suggests that general-purpose numerical convex programming algorithms can be used to solve for the saddle-point with polynomial complexity [31] [33] [34].

Finally, the main sum capacity result can be easily generalized to broadcast channels with an arbitrary convex input constraint. This is so because the achievability result using GDFE works with any arbitrary Gaussian input distribution, and the saddle-point for the mutual information expression is Gaussian as long as the input and noise constraints are convex. The generalization is stated as a corollary:

Corollary 1: Consider a Gaussian vector broadcast channel $\mathbf{y_i} = H_i\mathbf{x} + \mathbf{z_i}$, $i = 1, \cdots, K$, under a convex constraint $\mathcal{K}_x$. Let $H^T = [H_1^T \cdots H_K^T]$. The sum capacity is a saddle-point of the mutual information $\frac{1}{2}\log|HS_{xx}H^T + S_{zz}|/|S_{zz}|$ with the following constraints: S_{zz} has block-diagonal entries that are the covariance matrices of $\mathbf{z_1}, \cdots, \mathbf{z_K}$, and $\mathcal{N}(0, S_{xx})$ satisfies the input constraint $\mathcal{K}_x$.

V. Value of Cooperation

A principal aim of this paper is to illustrate the value of cooperation in a Gaussian vector channel $\mathbf{y} = H\mathbf{x} + \mathbf{z}$. When cooperation is possible both among the transmit terminals and among the receive terminals, the capacity of the vector channel under a power constraint is the solution to the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} \quad & \operatorname{tr}(S_{xx}) \le P, \\ & S_{xx} \ge 0. \end{aligned} \tag{114}$$

This leads to the well-known water-filling solution based on the singular-value decomposition of H [35]. Assuming that $S_{zz} = I$, then the optimum S_{xx} must have its eigenvectors equal to the right singular-vectors of H, and its eigenvalues obeying the water-filling power allocation on the singular-values of H. Further, the receive matrix can be chosen to match the left singular-vectors of H, so that the vector Gaussian channel is diagonalized into a series of independent scalar channels onto which single-user codes can be used to collectively achieve the vector channel capacity.

When coordination is possible only among the receive terminals, but not among the transmit terminals, the vector channel becomes a Gaussian multiple access channel. Although the sum capacity of a multiple access channel is still a maximum mutual information $I(\mathbf{X};\mathbf{Y})$, the transmit terminals of the multiple access channel must be uncorrelated. Thus, the water-filling covariance, which is optimum for a coordinated vector channel, can no longer necessarily be synthesized. The optimum covariance matrix for the multiple access channel must be found by solving an optimization problem that restricts the off-diagonal entries of the covariance matrix to zero.

$$\begin{aligned} \text{maximize} \quad & \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} \\ \text{subject to} \quad & \operatorname{tr}(S_{xx}) \le P, \\ & S_{xx}(i,j) = 0, \quad \forall (i,j) \text{ uncoordinated} \\ & S_{xx} \ge 0, \end{aligned} \tag{115}$$

where $S_{xx}(i,j)$ denotes the (i,j)-entry of S_{xx}. Thus, in terms of capacity, the value of cooperation at the transmitter lies in the ability for the transmitters to send correlated signals. In addition, the lack of transmitter coordination makes the diagonalization of the vector channel impossible. Instead, the vector channel can only be triangularized [23] [25]. Such triangularization decomposes a vector channel into a series of single-user sub-channels each interfering with only subsequent sub-channels. This enables a coding method based on the superposition of single-user codes and a decoding method based on successive decision-feedback to be implemented. The optimal form of triangularization is a generalized decision-feedback equalizer (GDFE). If decisions on previous sub-channels are assumed correct, GDFE achieves the sum capacity of a Gaussian vector multiple access channel [25].

When coordination is possible only among the transmit terminals, but not among the receive terminals, the vector channel becomes a Gaussian vector broadcast channel. The main result of this paper is that the sum capacity of a Gaussian vector broadcast channel is the saddle-point of a minimax problem

$$\begin{array}{ll} \max\limits_{S_{xx}} \min\limits_{S'_{zz}} & \frac{1}{2} \log \frac{|HS_{xx}H^T + S'_{zz}|}{|S'_{zz}|} \\ \text{subject to} & \mathrm{tr}(S_{xx}) \le P, \\ & S'_{zz}(i,j) = S_{zz}(i,j), \quad \forall (i,j) \text{ coordinated} \\ & S_{xx}, S_{zz} \ge 0. \end{array} \tag{116}$$

Because of the lack of coordination, the receivers can no longer distinguish between different noise correlations, and the capacity is as if "nature" has chosen a least favorable noise correlation. Thus, from a capacity point of view, the value of cooperation at the receiver lies in the ability for the receivers to recognize and to take advantage of the true correlation among the noise signals. Further, the result of this paper reveals that the structure of the sum-capacity achieving coding strategy for the Gaussian vector broadcast channel is a decision-feedback equalizer. The optimal coding strategy again decomposes the vector channel into independent scalar sub-channels each interfering into subsequent sub-channels, with the interference pre-subtracted using "writing-on-dirty-paper" coding. When full coordination is not possible, GDFE has emerged as a unifying structure that is capable of achieving the sum capacities of both the multiple access channel and the broadcast channel sum-capacity.

Appendix

The following numerical example illustrates the design of a precoder for the Gaussian vector broadcast channel. Consider the following channel:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1.0 & -0.3 & 0.2 \\ -0.4 & 2.0 & 0.5 \\ -0.1 & 0.2 & 3.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} + \begin{bmatrix} z_1 \\ z_2 \\ z_2 \end{bmatrix}, \tag{117}$$

where y_1, y_2, and y_3 are uncoordinated receivers, and z_1, z_2, and z_3 are i.i.d. Gaussian noises with variance 1. The total power constraint is set to 5. The iterative algorithm described at the end of the section 3.3 is used to solve for the saddle point (S_{xx}, S_{zz}). The water-filling step is standard. The least favorable noise problem is solved using an interior-point method. The algorithm converged in 3-4 steps. The numerical solution is:

$$S_{xx} = \begin{bmatrix} 1.0762 & -0.2327 & -0.0074 \\ -0.2327 & 1.8635 & 0.0387 \\ -0.0074 & 0.0387 & 2.0603 \end{bmatrix}, \quad S_{zz} = \begin{bmatrix} 1.0000 & -0.1286 & 0.0493 \\ -0.1286 & 1.0000 & 0.0311 \\ 0.0493 & 0.0311 & 1.0000 \end{bmatrix}. \tag{118}$$

To verify that the above solution satisfies the KKT conditions:

$$S_{zz}^{-1} - \left(S_{zz} + HS_{xx}H^T\right)^{-1} = \Psi = \begin{bmatrix} 0.4859 & 0 & 0 \\ 0 & 0.8701 & 0 \\ 0 & 0 & 0.9422 \end{bmatrix} \tag{119}$$

and

$$H^T\left(HS_{xx}H^T + S_{zz}\right)^{-1} H = \begin{bmatrix} 0.4597 & 0 & 0 \\ 0 & 0.4597 & 0 \\ 0 & 0 & 0.4597 \end{bmatrix}. \tag{120}$$

The vector channel capacity with the least favorable noise correlation is:

$$C = \frac{1}{2}\log\frac{|HS_{xx}H^T + S_{zz}|}{|S_{zz}|} = 2.8952. \tag{121}$$

The objective is to design a generalized decision-feedback precoder that achieves the vector channel capacity without receiver coordination. This is accomplished by finding an appropriate transmit filter $B = V\Sigma^{\frac{1}{2}}M$ which would induce a diagonal feedforward filter in a GDFE. Following the proof of Lemma 4, compute the eigen-decomposition $S_{xx} = V\Sigma V^T$ and $S_{zz} = Q^T\Lambda Q$. Then, compute R as a square root of the following as in (79):

$$R^T R = \left(\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} + I\right)^{-1}. \tag{122}$$

In particular, R can be found by a Cholesky factorization. In this example, because S_{xx} is the water-filling covariance, the matrix V diagonalizes the channel, so that $R^T R$ is already diagonal. So, finding an R is trivial. Numerically,

$$R = \begin{bmatrix} 0.2191 & 0 & 0 \\ 0 & 0.3451 & 0 \\ 0 & 0 & 0.7312 \end{bmatrix}. \tag{123}$$

The next step is to find an orthogonal matrix U, such that $UR\sqrt{\Sigma}V^T H^T Q^T \Lambda^{-1} Q$ is diagonal. The proof of Lemma 4 shows that U can be found as follows:

$$U = \Psi^{-\frac{1}{2}} Q^T \Lambda^{-1} Q H V \sqrt{\Sigma} R^T = \begin{bmatrix} 0.0115 & -0.2220 & 0.9750 \\ 0.3147 & 0.9263 & 0.2072 \\ 0.9491 & -0.3045 & -0.0805 \end{bmatrix}. \tag{124}$$

The final step is to find an orthogonal matrix M such that URM is lower-triangular. This is done by computing the QR-factorization of $R^TU^T = MT$, where T is upper-triangular, and M is orthogonal. Then, $URM = T^T$ is lower-triangular. In this example,

$$R^TU^T = MT = \begin{bmatrix} -0.0035 & -0.2010 & -0.9796 \\ 0.1069 & -0.9740 & 0.1995 \\ -0.9943 & -0.1040 & 0.0249 \end{bmatrix} \begin{bmatrix} -0.7170 & -0.1167 & 0.0466 \\ 0 & -0.3410 & 0.0666 \\ 0 & 0 & -0.2262 \end{bmatrix}. \quad (125)$$

This gives the appropriate M for the desired transmit filter $B = V\Sigma^{\frac{1}{2}}M$.

Now, design a generalized decision-feedback equalizer for the effective channel

$$\widetilde{H} = \frac{1}{\sqrt{\Lambda}}QHV\sqrt{\Sigma}M = \begin{bmatrix} -0.7439 & 2.2489 & 0.0128 \\ 0.1698 & 0.8505 & 4.3105 \\ 0.6027 & 1.4311 & -0.8596 \end{bmatrix}, \quad (126)$$

an input covariance $S_{uu} = I$, and a noise covariance $S_{zz} = I$. Compute $G^{-1}\Delta^{-1}G^{-T} = (\widetilde{H}^T\widetilde{H}+I)^{-1}$:

$$G = \begin{bmatrix} 1 & -0.3423 & 0.1051 \\ 0 & 1 & 0.2947 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1.9454 & 0 & 0 \\ 0 & 8.6009 & 0 \\ 0 & 0 & 19.5512 \end{bmatrix}. \quad (127)$$

As expected, the choice of transmit filter makes the feedforward filter a diagonal matrix:

$$F = \Delta^{-1}G^{-T}\widetilde{H}^T\frac{1}{\sqrt{\Lambda}}Q = \begin{bmatrix} -0.4998 & 0 & 0 \\ 0 & -0.3181 & 0 \\ 0 & 0 & -0.2195 \end{bmatrix}. \quad (128)$$

First, let's compute the capacities of individual sub-channels in the GDFE feedback configuration. The effective channel is $\mathbf{u}' = FHB\mathbf{u} + (I - G)\mathbf{u} + F\mathbf{z}$:

$$\begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 0.4860 & 0 & 0 \\ -0.0398 & 0.8837 & 0 \\ -0.0105 & 0.0151 & 0.9489 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} -0.4998z_3 \\ -0.3181z_2 \\ -0.2195z_1 \end{bmatrix}. \quad (129)$$

Thus, the capacities of the three sub-channels are:

$$R_1 = \frac{1}{2}\log\left(1 + \frac{0.4860^2}{0.4998^2}\right) = 0.3327 \quad (130)$$

$$R_2 = \frac{1}{2}\log\left(1 + \frac{0.8837^2}{0.3181^2 + 0.0398^2}\right) = 1.0759 \quad (131)$$

$$R_3 = \frac{1}{2}\log\left(1 + \frac{0.9489^2}{0.0105^2 + 0.0151^2 + 0.133^2}\right) = 1.4865. \quad (132)$$

The sum capacity is $R_1 + R_2 + R_3 = 2.8952$, which agrees with the vector channel capacity.

Now, compute the capacity of individual sub-channels in the precoding configuration. The effective channel is $\mathbf{y} = HB\mathbf{u} + \mathbf{z}$:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -0.9723 & 0.6847 & -0.2101 \\ 0.1251 & -2.7785 & -0.9265 \\ -0.0480 & -0.0687 & -4.3222 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} z_3 \\ z_2 \\ z_1 \end{bmatrix}. \quad (133)$$

Decoding u_3 from y_3, the capacity is:

$$R_3 = \frac{1}{2}\log\left(1 + \frac{4.3222^2}{1 + 0.0480^2 + 0.0687^2}\right) = 1.4865. \quad (134)$$

The signal from u_3 may be pre-subtracted from u_2, leading to:

$$R_2 = \frac{1}{2}\log\left(1 + \frac{2.7785^2}{1 + 0.1251^2}\right) = 1.0759. \quad (135)$$

The signals from u_2 and u_3 may be pre-subtracted from u_1, leading to:

$$R_1 = \frac{1}{2}\log(1 + 0.9723^2) = 0.3327. \quad (136)$$

Therefore, without receiver coordination, a sum capacity of $R_1 + R_2 + R_3 = 2.8952$ is also achievable. In fact, it is now possible to identify the appropriate transmit covariance matrices for each user as in Theorem 1. Let B_1, B_2 and B_3 be the column vectors of the transmit filter $B = [B_1 B_2 B_3]$. Then information bits u_1, u_2 and u_3 are modulated with covariance matrices $S_1 = B_1 B_1^T$, $S_2 = B_2 B_2^T$ and $S_3 = B_3 B_3^T$. Let H_1, H_2 and H_3 be the row vectors of the channel $H^T = [H_1^T H_2^T H_3^T]$. Then, by Theorem 1, the following rates are achievable:

$$R_1 = \frac{1}{2}\log\left(H_1 S_1 H_1^T + 1\right) = 0.3327 \quad (137)$$

$$R_2 = \frac{1}{2}\log\left(\frac{H_2 S_2 H_2^T + H_2 S_1 H_2^T + 1}{H_2 S_1 H_2^T + 1}\right) = 1.0759 \quad (138)$$

$$R_3 = \frac{1}{2}\log\left(\frac{H_3 S_3 H_3^T + H_3 S_2 H_3^T + H_3 S_1 H_3^T + 1}{H_3 S_2 H_3^T + H_3 S_1 H_3^T + 1}\right) = 1.4865. \quad (139)$$

This again verifies that $R_1 + R_2 + R_3 = 2.8952$ is achievable with no coordination at the receiver side.

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