# Duality and the Value of Cooperation in Distributive Source and Channel Coding

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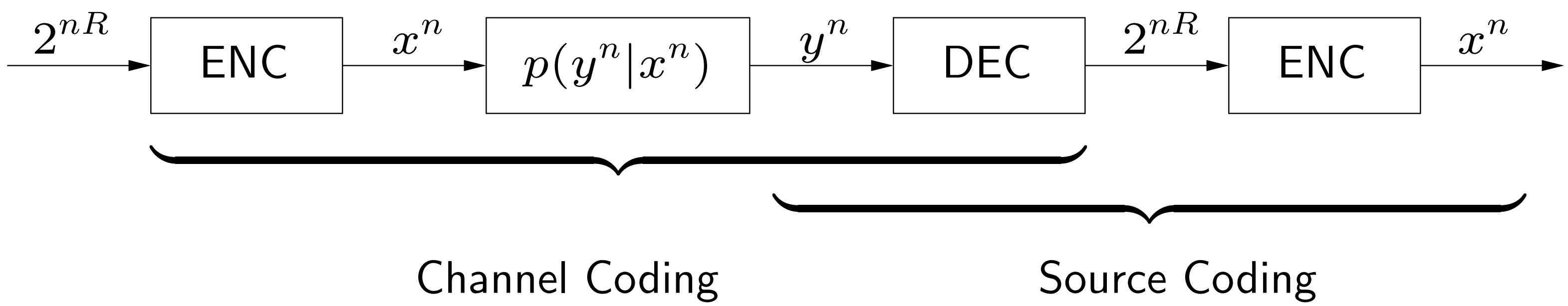
# Source and Channel Coding Duality

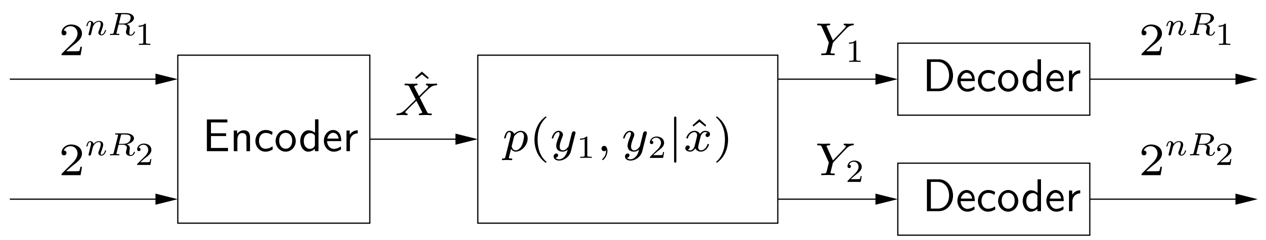
- There is a duality between source and channel coding (Shannon 1959):
  - Fix a joint distribution $p(x, y)$:
  - $p(y|x)$ becomes a test channel; Encoder $\Longleftrightarrow$ Decoder.




**Does duality extend to multi-user channels?**

# Broadcast Channel

- Joint encoder, separate decoder:

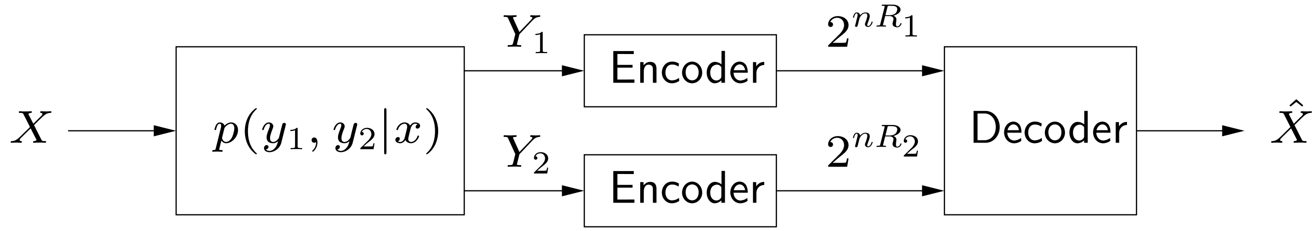



  - Solution known for degraded and deterministic broadcast channels. (Cover 71, Gallager 73, Bergmans 73, Pinsker, Gel'fand 79, Marton 79.)

General solution unknown.

# Distributive Source Coding

- Separate encoders, joint decoder:

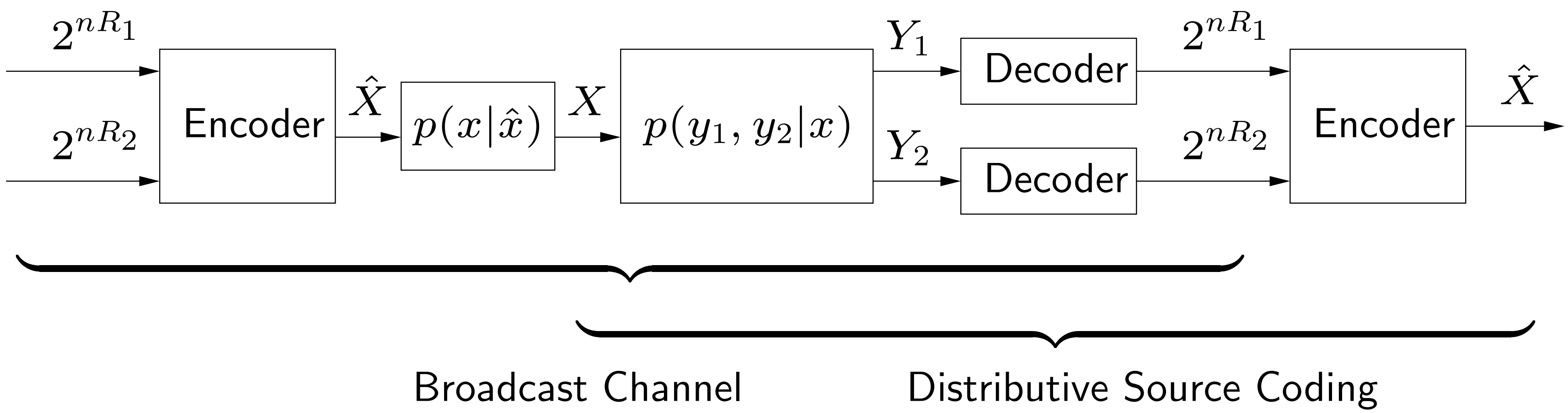



  - Direct coding: Decoder estimates both $Y_1$ and $Y_2$.
    (Berger, Viswanathan, 96, 97, Oohama 97, 98. Zamir, Berger, 99)
  - Indirect coding: Decoder estimates an underlying $X$.
    (Oohama 98. Zamir, Berger, 99)

Known as the CEO problem. General solution unknown.

# Duality of Distributive Source and Channel Coding

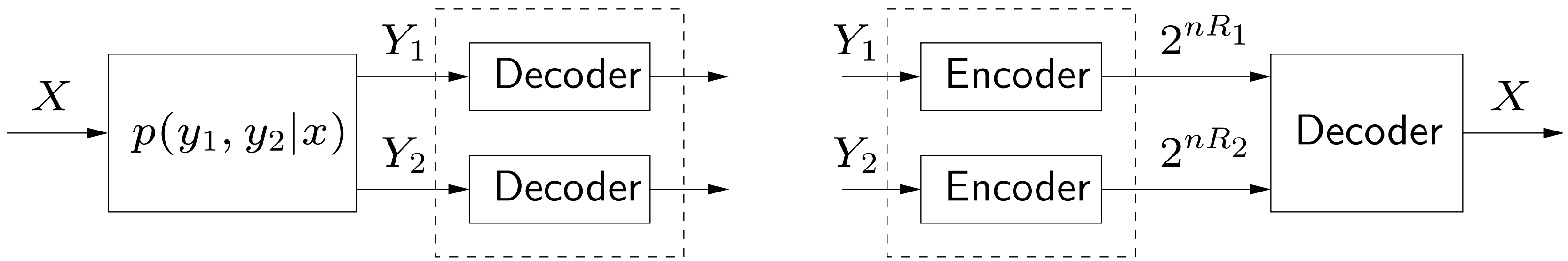
- Duality: Encoder $\Longleftrightarrow$ Decoder. $p(x|\hat{x})p(y_1, y_2|x)$ is the test channel.




Does multi-user source-channel duality exist?

# Value of Cooperation

- Cooperative encoders/decoders usually do better:




Broadcast Channel                Distributive Source Coding

**When do non-cooperative encoders/decoders do as well?**

# Main Result

| Duality | $\Longleftrightarrow$ | No Rate Loss<br>in<br>Non-Cooperative Coding |
|---|---|---|

There are interesting examples for which duality holds.

# Previous Work on Duality

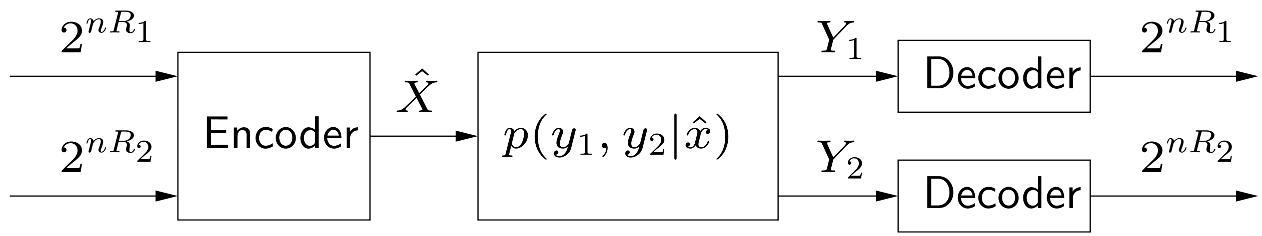
- Pradhan, Ramchandran (ITW '02) considered "direct coding" problem
  - Duality exists iff the same joint distribution suits both problems.
  - This happens iff two simultaneous Markov chains are satisfied.

- Viswanath (DIMACS '03) considered "indirect coding" problem
  - Constructed a specific Gaussian example.

- This paper considers the general "indirect coding" problem.
  - Duality exists iff cooperative coding achieves the same sum rate as with no cooperation
  - We give a novel discrete memoryless channel example for which cooperation doesn't help.

# Marton's Region for Broadcast Channel

- Discrete memoryless broadcast channel with Marton's binning strategy:

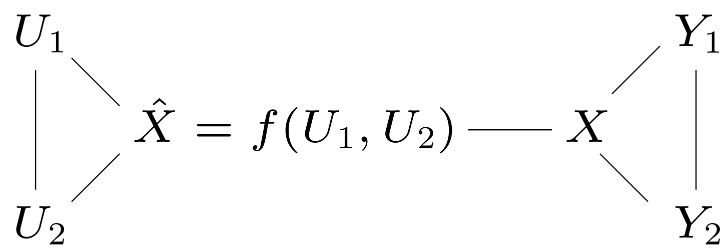



The following is achievable for any $p(u_1, u_2)p(\hat{x}|u_2, u_2)p(y_1, y_2|\hat{x})$:

$$
\begin{aligned}
R_1 &\leq I(U_1; Y_1); \\
R_2 &\leq I(U_2; Y_2); \\
R_1 + R_2 &\leq I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2).
\end{aligned}
$$

# Markov Condition for Marton's Region

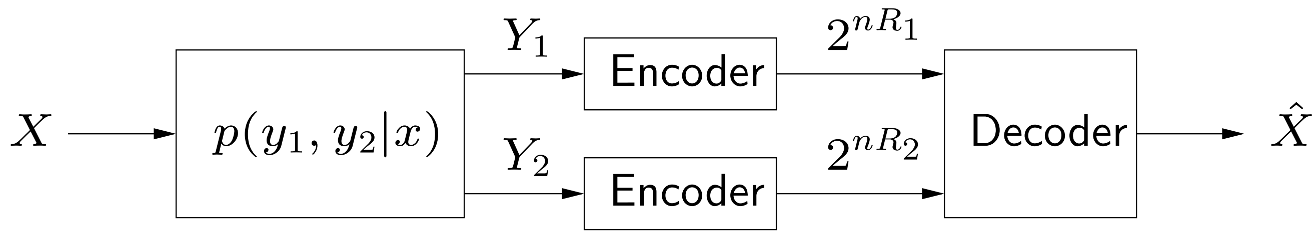
- The joint distribution is of the form: $p(u_1, u_2)p(\hat{x}|u_2, u_2)p(y_1, y_2|\hat{x})$.




It must satisfy the Markov condition: $(U_1, U_2) - X - \hat{X} - (Y_1, Y_2)$.

# Berger-Tung's Region for Distributive Source Coding

- Extending Berger-Tung's binning strategy to indirect coding:

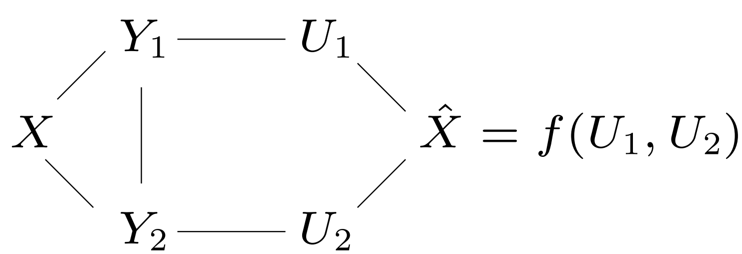



**Theorem 1.** *The following rate region is achievable for any fixed distribution* $p(x)p(y_1, y_2|x)p(u_1|y_1)p(u_2|y_2)p(\hat{x}|u_2, u_2)$*:*

$$
\begin{aligned}
R_1 &\geq I(U_1, Y_1|U_2); \\
R_2 &\geq I(U_2, Y_2|U_1); \\
R_1 + R_2 &\geq I(U_1, U_2; Y_1, Y_2),
\end{aligned}
$$

# Markov Condition for Berger-Tung

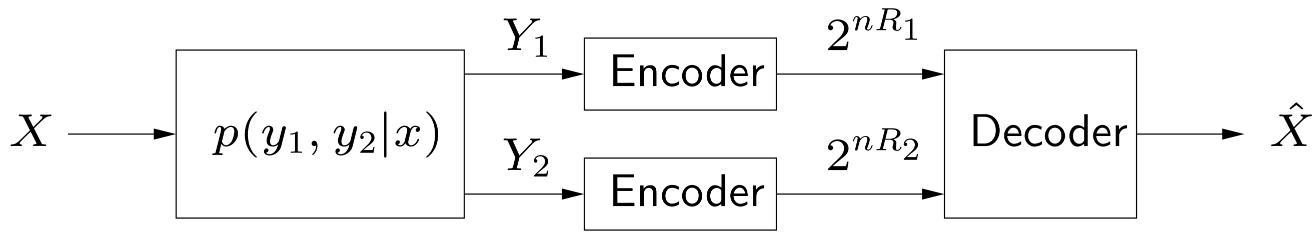
- The joint distribution is $p(x)p(y_1, y_2|x)p(u_1|y_1)p(u_2|y_2)p(\hat{x}|u_2, u_2)$.




It must satisfy the Markov condition: $U_1 - Y_1 - Y_2 - U_2$.

# An Alternative Expression for Berger-Tung Region

- Using the Markov chain $U_1 - Y_1 - Y_2 - U_2$:

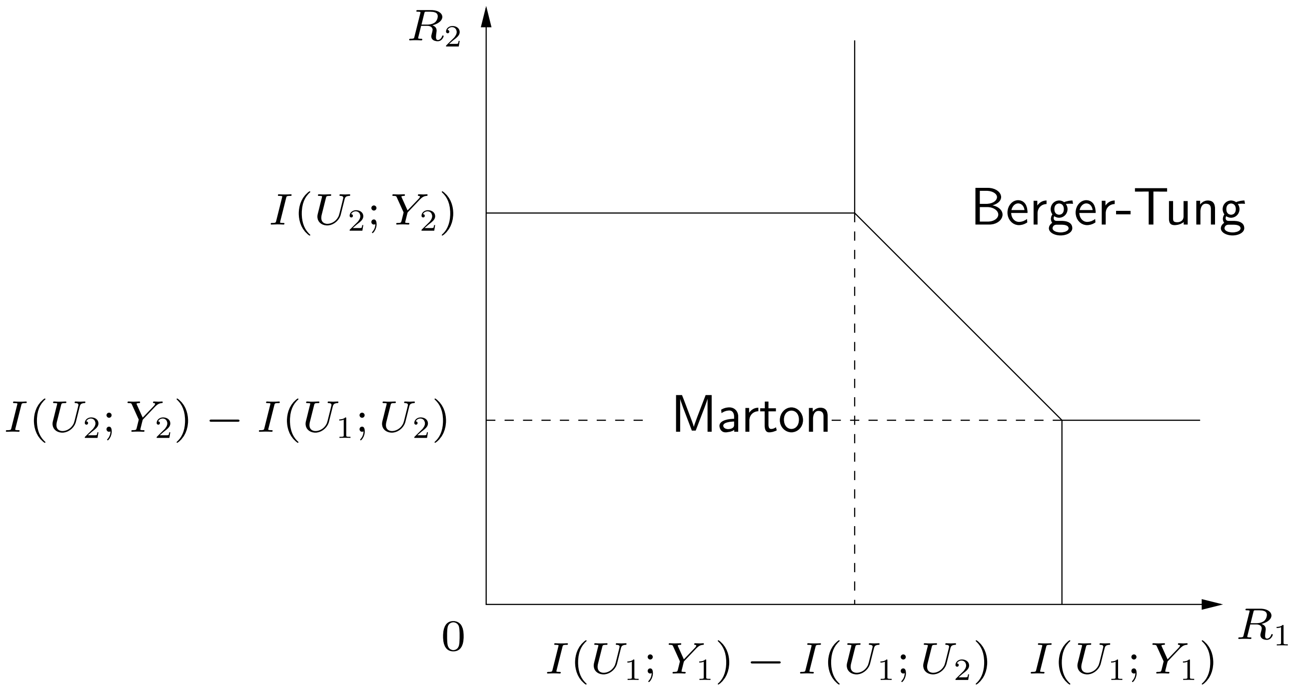



**Theorem 2.** *The following rate region is achievable for any fixed distribution* $p(x)p(y_1, y_2|x)p(u_1|y_1)p(u_2|y_2)p(\hat{x}|u_2, u_2)$*:*

$$
\begin{aligned}
R_1 &\geq I(U_1; Y_1) - I(U_1; U_2); \\
R_2 &\geq I(U_2; Y_2) - I(U_1; U_2); \\
R_1 + R_2 &\geq I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2),
\end{aligned}
$$

# Duality of the Rate Region

**Definition 1.** *If a joint distribution* $p(\hat{x}, x, y_1, y_2, u_1, u_2)$ *satisfies both Markov chain conditions, then their respective distributive source coding and channel coding problems are duals of each other.*



# Rate Loss in Distributive Coding

- Broadcast channel:
  - Cooperative decoder can potentially achieve higher rate:

$$I(\hat{X};Y_1,Y_2) \geq I(U_1;Y_1) + I(U_2;Y_2) - I(U_1;U_2)$$

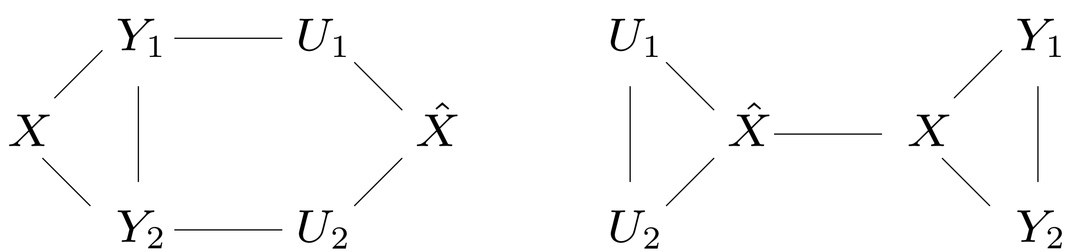
- Distributive source coding:
  - Cooperative encoder can potentially achieve lower rate:

$$I(X;\hat{X}) \leq I(U_1;Y_1) + I(U_2;Y_2) - I(U_1;U_2)$$

# Duality Implies No Rate Loss

**Theorem 3.** *If duality holds, then in both broadcast channel and distributive source coding problems joint coding achieves the same sum rate as separate coding.*

Proof: Both Markov chains are satisfied, so: $I(X;\hat{X}) = I(U_1, U_2; Y_1, Y_2)$.




For a distributive source coding problem, $I(X;\hat{X})$ is the lower bound. But, $I(U_1, U_2; Y_1, Y_2)$ is achievable. So, there is no rate loss.

Similar argument holds for the broadcast channel.

# No Rate Loss Implies Duality (1)

**Theorem 4. [Viswanath]** *If the sum rate of a broadcast channel achieves its cooperative upper bound, then it has a dual distributive source coding problem in which the sum rate achieves its cooperative lower bound.*

Proof: If broadcast channel achieves $I(\hat{X}; Y_1, Y_2)$, then, since

$$\begin{aligned} I(\hat{X}; Y_1, Y_2) &\geq I(U_1, U_2; Y_1, Y_2) \\ &= I(U_1; Y_1, Y_2) + I(U_2; Y_1, Y_2, U_1) - I(U_1; U_2) \\ &\geq I(U_1; Y_1) + I(U_2; Y_2) - I(U_1; U_2), \end{aligned}$$

it must be that $(U_1, Y_1) - Y_2 - U_2$ and $U_1 - Y_1 - Y_2$. Therefore,

$$U_1 - Y_1 - Y_2 - U_2.$$

# No Rate Loss Implies Duality (2)

**Theorem 5.** *If the sum rate of a distributive source coding problem achieves its cooperative lower bound, then it has a dual broadcast channel with a sum rate achieving its cooperative upper bound.*

**Lemma 1.** *If $X - Y - Z$ forms a Markov chain and $I(X;Z) = I(Y;Z)$, then $Y - X - Z$ also forms a Markov chain.*

Proof of the Theorem: Since the source coding problem has no rate loss, then $I(\hat{X};X) = I(U_1, U_2; Y_1, Y_2)$. By source coding Markov chain: $X - (Y_1, Y_2) - (U_1, U_2) - \hat{X}$, and repeatedly applying Lemma, we get

$$(U_1, U_2) - \hat{X} - X - (Y_1, Y_2).$$

This defines a dual broadcast channel problem, and it has no rate loss.

# Duality $\Longleftrightarrow$ No Rate Loss

- Fix a joint distribution:
  - Duality $\Longrightarrow$ Two Markov Chains $\Longrightarrow$ No rate loss.
  - No rate loss $\Longrightarrow$ Two Markov Chains $\Longrightarrow$ Duality.

- Disclaimer: Duality does not necessarily imply that

$$\max_{p(u_1,u_2,x)} I(U_1;Y_1) + I(U_2;Y_2) - I(U_1;U_2) = \max_{p(x)} I(X;Y_1,Y_2)$$

  - "Duality implies no rate loss" is for a fixed distribution only.

# Examples

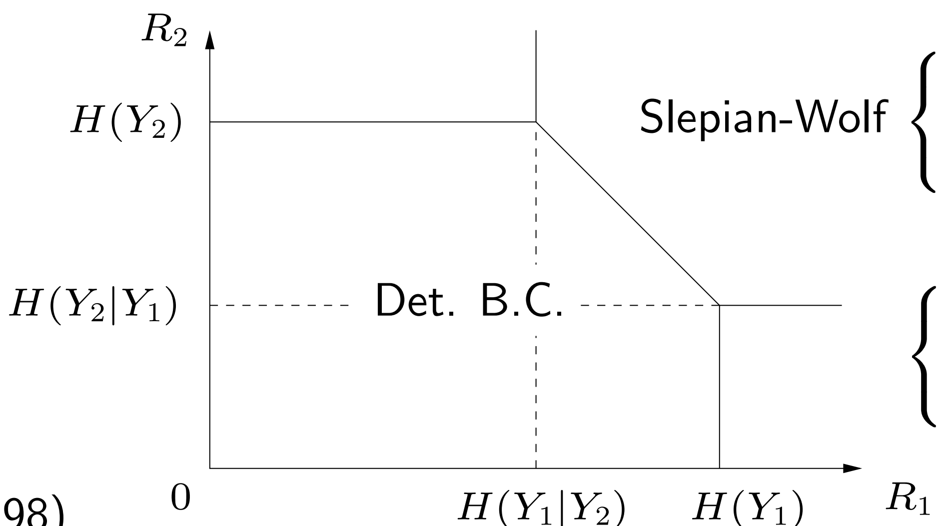
- Deterministic channels (Cover '98)
- Gaussian vector channels (Viswanath '03)
- A discrete memoryless channel based on memory with defects.

# Deterministic Channel: Example (1)

- Deterministic broadcast channel: $Y_1 = f_1(X)$, $Y_2 = f_2(X)$.
- Slepian-Wolf problem: $\boxed{R_1 + R_2 = H(Y_1, Y_2)}$.

$U_1 = Y_1$

$U_2 = Y_2$

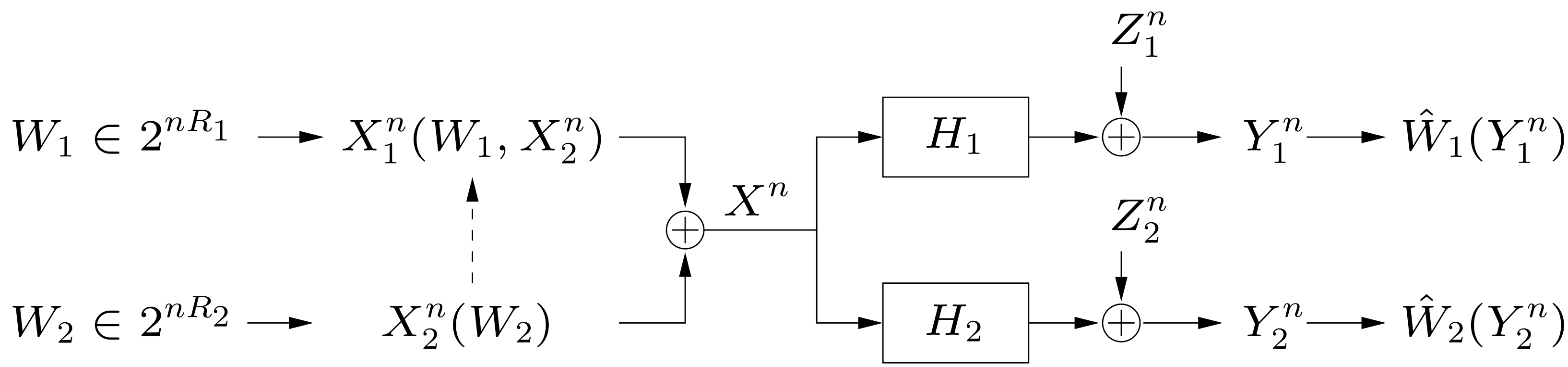



Slepian-Wolf $\begin{cases} R_1 \geq H(Y_1|Y_2) \\ R_2 \geq H(Y_2|Y_1) \\ R_1 + R_2 \geq H(Y_1, Y_2) \end{cases}$

$$\begin{cases} R_1 \leq H(Y_1) \\ R_2 \leq H(Y_2) \\ R_1 + R_2 \leq H(Y_1, Y_2) \end{cases}$$

(Cover '98)

# Gaussian Channel: Example (2)




- Dirty-paper coding:

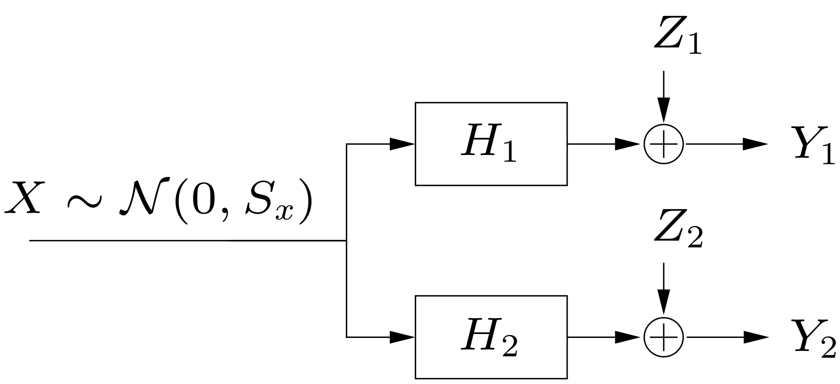
$$
\begin{aligned}
R_1 &= I(X_1; Y_1 | X_2); \\
R_2 &= I(X_2; Y_2).
\end{aligned}
$$

# Sum Capacity

- Fix $S_x$, $I(X;Y_1,Y_2)$ is achievable with the worst-noise:

$$\min_{S_z} \quad \frac{1}{2}\log\frac{|HS_xH^T+S_z|}{|S_z|}$$

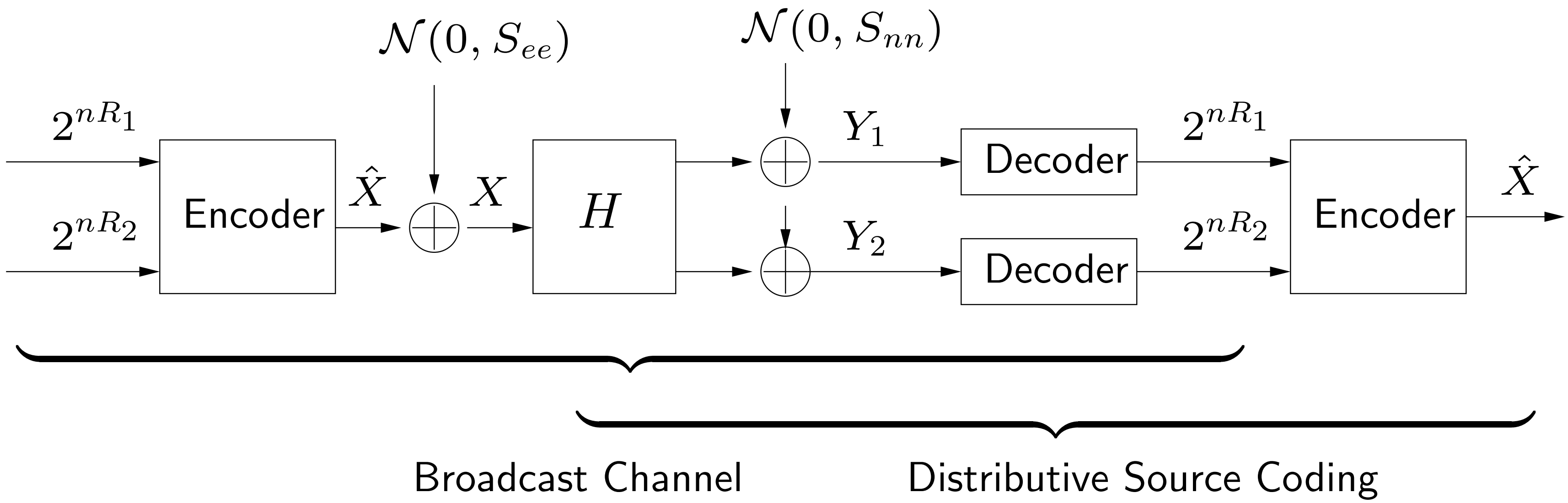
$$\text{s.t.} \quad S_z = \begin{bmatrix} I & \star \\ \star & I \end{bmatrix}$$




$R_1 + R_2 = I(X;Y_1,Y_2)$, with worst-noise.

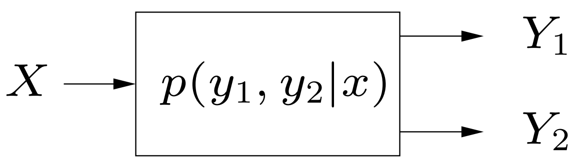
# Source-Channel Duality in Gaussian Channels

- Duality exists iff $S_{nn} + HS_{ee}H^T$ is the worst-noise.

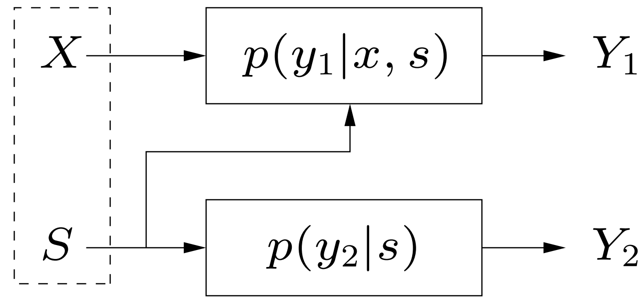



(This generalizes Viswanath's result, where $S_{nn} = 0$, $S_{xx} = I$.)

# Discrete Memoryless Channel: Example (3)




Broadcast Channel

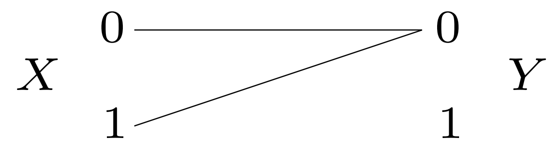



Channel with Side Information

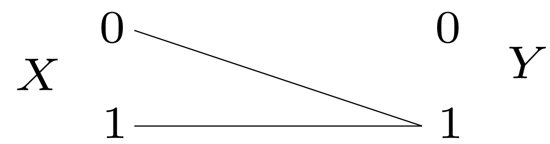
- When is $I(X; Y_1, Y_2)$ achievable in a broadcast channel?
- Idea: Construct a broadcast channel from channels with side information
  - Hope: $I(U; Y_1) - I(U; S) = I(X; Y_1|S) \Rightarrow$ no rate loss.

# Memory with Defect

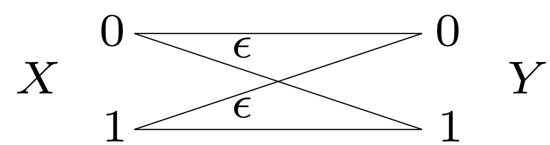
$S = 0$: Stuck-at-zero $\left(\frac{p}{2}\right)$




$S = 1$: Stuck-at-one $\left(\frac{p}{2}\right)$



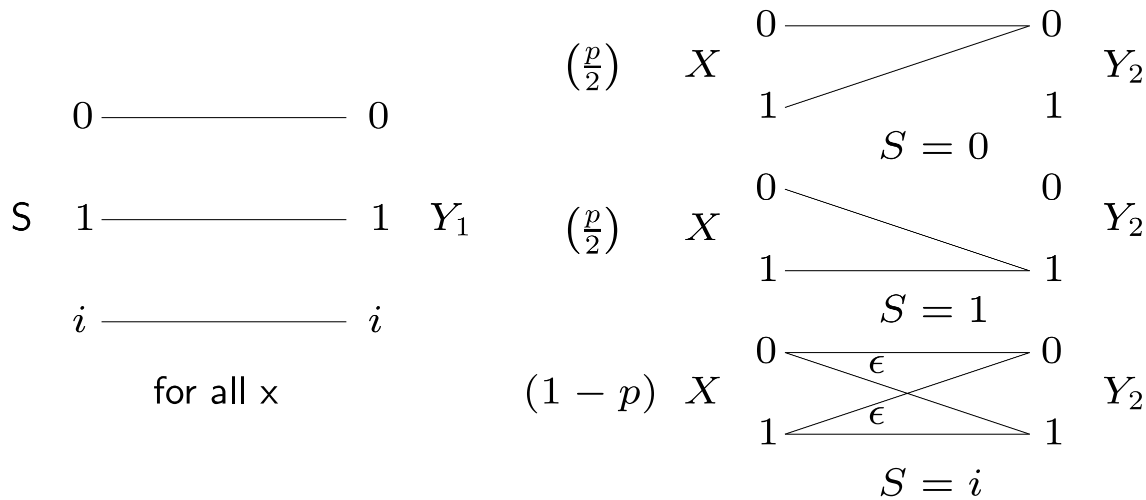

$S = i$: Binary Symmetric Channel $(1 - p)$




- Set $U = X$. Then, $I(U;Y) - I(U;S) = I(X;Y|S)$
  - Same capacity with or without state information at the receiver.
  - Heegard and El Gamal ('83)

# Write-Once Memory

- How to use "write-once" memory twice?

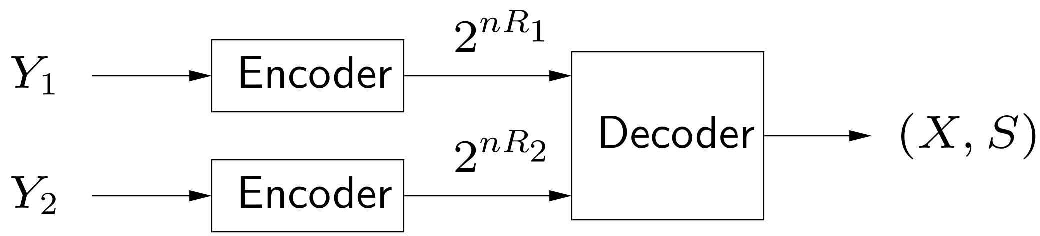



  - Write once to communicate with $Y_1$, twice to communicate with $Y_2$.

$$I(X, S; Y_1, Y_2) = H(S) + I(X; Y_2|S)$$

# Duality for Write-Once Memory

- The "Write-once" broadcast channel has one deterministic component.

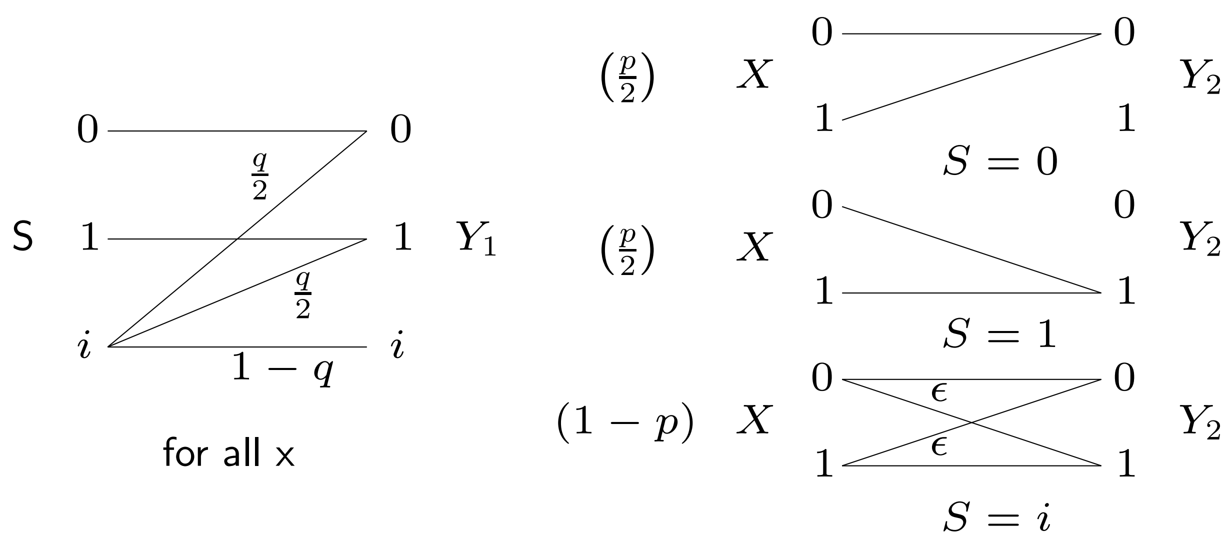



$$Y_1 = S,\ R_1 = H(S),\ R_2 = I(Y_2, X|S)$$

- The dual distributive source coding problem is a Wyner-Ziv problem.

# Write-Once Memory with Imperfect State Information

- The first receiver $Y_1$ only gets an imperfect $S$:




- Is it still true that $I(X,S;Y_1,Y_2) = I(S;Y_1) + I(X;Y_2|S)$?

# Condition for No Rate Loss for Write-Once Memory

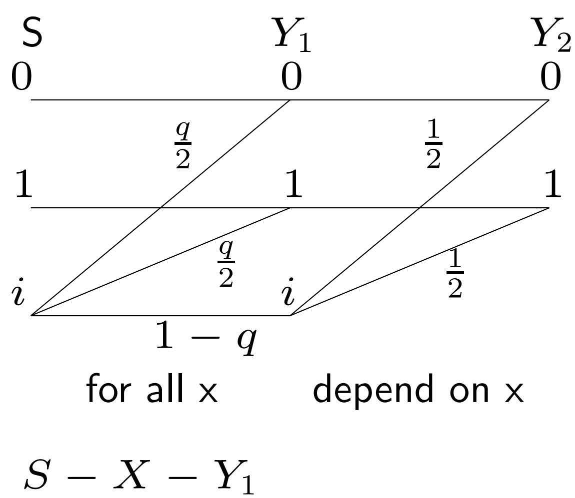
- Want $I(X,S;Y_1,Y_2) = I(S;Y_1) + I(X;Y_2|S)$. But,

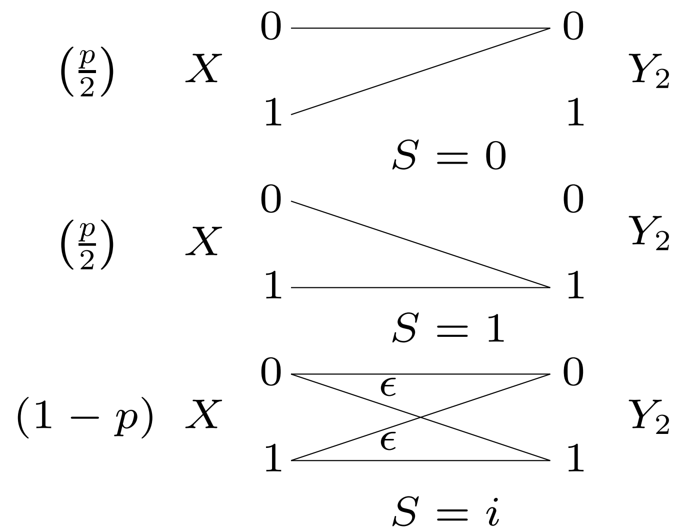
$$\begin{aligned} I(X,S;Y_1,Y_2) &= I(X,S;Y_1) + I(X,S;Y_2|Y_1) \\ &= I(S;Y_1) + I(X;Y_1|S) + I(S;Y_2|Y_1) + I(X;Y_2|Y_1,S). \end{aligned}$$

- Need:
  - $X - S - Y_1$.
  - $S - Y_1 - Y_2$.
  - $I(X;Y_2|Y_1,S) = I(X;Y_2|S)$.

# Revised Write-Once Broadcast Channel

- Need to choose a worst noise distribution $p(y_1, y_2|x, s)$.




- In addition, choose $p(y_2|y_1, x)$ to give $I(X; Y_2|Y_1, S) = I(X; Y_2|S)$. (Possible only for specific values of $\epsilon$ and $q$.)

# Duality Exists for Some Write-Once Channels

- For such specifically designed "write-once" channel:

$$\begin{aligned} R_1 &= I(S; Y_1) \\ R_2 &= I(X; Y_2 | S) \\ R_1 + R_2 &= I(X, S; Y_1, Y_2) \end{aligned}$$

- There is no rate loss with respect to cooperative broadcasting.

- A dual distributive source coding problem exists, also with no rate loss.

# Conclusion

- Duality exists between broadcast channel and distributive source coding
  - Duality holds if and only if there is no rate loss with respect to cooperative coding.
- Examples of duality:
  - Deterministic channel.
  - Gaussian vector broadcast channel.
  - Write-once memory channel.
- In general, cases of duality are few and far between.