1. **Differential Entropy**
   Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:
   
   (a) The Laplace density, $f(x) = \frac{1}{2} \lambda e^{-\lambda |x|}$.
   (b) The sum of $X_1$ and $X_2$, where $X_1$ and $X_2$ are independent normal random variables with means $\mu_i$ and variances $\sigma^2_i, i = 1, 2$.

2. **Gaussian Mutual Information**
   Suppose that $(X, Y, Z)$ are jointly Gaussian and that $X \rightarrow Y \rightarrow Z$ forms a Markov chain. Let $X$ and $Y$ have correlation coefficient $\rho_1$ and let $Y$ and $Z$ have correlation coefficient $\rho_2$. Find (a) $I(X; Y)$, (b) $I(Y; Z)$, (c) $I(X; Z)$.

3. **Additive noise channel.**
   Consider the channel $Y = X + Z$, where $X$ is the transmitted signal with power constraint $P$, $Z$ is independent additive noise, and $Y$ is the received signal. Let
   
   $$Z = \begin{cases} 
   0 & \text{with probability } \frac{1}{10}, \\
   Z^* & \text{with probability } \frac{9}{10},
   \end{cases}$$
   
   where $Z^* \sim N(0, N)$. Thus, $Z$ has a mixture distribution that is the mixture of a Gaussian distribution and a degenerate distribution with mass 1 at 0.
   
   (a) What is the capacity of this channel? This should be a pleasant surprise.
   (b) How would you signal to achieve capacity?

4. **Uniformly distributed noise.**
   The input and the output of the channel are related as $Y = X + Z$, where the noise random variable is uniformly distributed over the interval $-\frac{a}{2} \leq z \leq +\frac{a}{2}$.
   
   (a) Let the input random variable $X$ be uniformly distributed over the interval $-\frac{1}{2} \leq x \leq +\frac{1}{2}$. Find $I(X; Y)$ as a function of $a$.
   (b) For $a = 1$ find the capacity of the channel when the input $X$ is peak-limited; that is, the range of $X$ is limited to $-\frac{1}{2} \leq x \leq +\frac{1}{2}$. What probability distribution on $X$ maximizes the mutual information $I(X; Y)$?
   (c) Find the capacity of the channel for all values of $a$, again assuming that the range of $X$ is limited to $-\frac{1}{2} \leq x \leq +\frac{1}{2}$.
5. **Exponential noise channels.**

\[ Y_i = X_i + Z_i, \]  
where \( Z_i \) is i.i.d. exponentially distributed noise with mean \( \mu \). Assume that we have a mean constraint on the signal (i.e., \( EX_i \leq \lambda \)). Show that the capacity of such a channel is \( C = \log(1 + \frac{\lambda}{\mu}) \).

6. **Fading Channel**

Consider an additive noise fading channel

\[
\begin{align*}
V & \quad Z \\
\downarrow & \quad \downarrow \\
X & \quad \oplus \quad Y \\
\end{align*}
\]

\[ Y = XV + Z, \]

where \( Z \) is additive noise, \( V \) is a random variable representing fading, and \( Z \) and \( V \) are independent of each other and of \( X \). Argue that knowledge of the fading factor \( V \) improves capacity by showing

\[ I(X;Y|V) \geq I(X;Y). \]

7. **The Two-Look Gaussian Channel**

Consider the ordinary additive noise Gaussian channel with two correlated looks at \( X \), i.e.,

\[
Y = (Y_1, Y_2), \quad Y_1 = X + Z_1, \quad Y_2 = X + Z_2
\]

with a power constraint \( P \) on \( X \), and \( (Z_1, Z_2) \sim \mathcal{N}_2(0, K) \), where

\[
K = \begin{bmatrix}
N & N\rho \\
N\rho & N
\end{bmatrix}
\]

Find the capacity \( C \) for \( (a) \ \rho = 1, (b) \ \rho = 0, (c) \ \rho = -1. \)

8. **Parallel Channels And Waterfilling**

Consider a pair of parallel Gaussian channels, i.e.,

\[
\begin{pmatrix}
Y_1 \\
Y_2
\end{pmatrix}
= \begin{pmatrix}
X_1 \\
X_2
\end{pmatrix} + \begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
Z_1 \\
Z_2
\end{pmatrix}
\sim \mathcal{N} \left( 0, \begin{bmatrix}
\sigma_1^2 & 0 \\
0 & \sigma_2^2
\end{bmatrix} \right)
\]

and there is a power constraint \( E(X_1^2 + X_2^2) \leq P \). Assume that \( \sigma_1^2 > \sigma_2^2 \). At what power does the channel stop behaving like a single channel with noise variance \( \sigma_2^2 \), and begin behaving like a pair of channels, i.e., at what power does the worst channel become useful?
9. **Vector Gaussian channel.**
Consider the vector Gaussian noise channel

\[ Y = X + Z, \]

where \( X = (X_1, X_2, X_3) \), \( Z = (Z_1, Z_2, Z_3) \), \( Y = (Y_1, Y_2, Y_3) \), \( E\|X\|^2 \leq P \), and

\[ Z \sim \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{pmatrix}. \]

Find the capacity. The answer may be surprising.

10. **A Mutual Information Game**
Consider the following channel:

\[ \begin{array}{c}
X \\
+ \\
Y
\end{array} \]
\[ Z \]

Throughout this problem we shall constrain the signal power

\[ EX = 0, \quad EX^2 = P, \]

and the noise power

\[ EZ = 0, \quad EZ^2 = N, \]

and assume that \( X \) and \( Z \) are independent. The channel capacity is given by \( I(X; X + Z) \).

Now for the game. The noise player chooses a distribution on \( Z \) to minimize \( I(X; X + Z) \), while the signal player chooses a distribution on \( X \) to maximize \( I(X; X + Z) \).

Let \( X^* \sim \mathcal{N}(0, P); Z^* \sim \mathcal{N}(0, N) \). Show that Gaussian \( X^* \) and \( Z^* \) satisfy the saddlepoint conditions

\[ I(X; X + Z^*) \leq I(X^*; X^* + Z^*) \leq I(X^*; X^* + Z). \]

Thus

\[ \min_X \max_Z I(X; X + Z) = \max_X \min_Z I(X; X + Z) = \frac{1}{2} \log \left( 1 + \frac{P}{N} \right), \]

and the game has a value. In particular, a deviation from normal for either player worsens the mutual information from that player’s standpoint. Can you discuss the implications?

**Note:** Part of the proof hinges on the entropy power inequality, which states that if \( X \) and \( Y \) are independent random \( n \)-vectors with densities, then

\[ e^{\frac{2}{n} h(X + Y)} \geq e^{\frac{2}{n} h(X)} + e^{\frac{2}{n} h(Y)}. \]