FDMA Capacity of Gaussian Multiple-Access Channels With ISI

Wei Yu , *Student Member, IEEE,* and John M. Cioffi, *Fellow, IEEE*

***Abstract*—This paper proposes a numerical method for characterizing the rate region achievable with frequency-division multiple access (FDMA) for a Gaussian multiple-access channel with intersymbol interference. The frequency spectrum is divided into discrete frequency bins and the discrete bin-assignment problem is shown to have a convex relaxation, making it tractable to numerical optimization algorithms. A practical low-complexity algorithm for the two-user case is also proposed. The algorithm is based on the observation that the optimal frequency partition has a two-band structure when the two channels are identical or when the signal-to-noise ratio is high. Simulation result shows that the algorithm performs well in other cases as well. The FDMA-capacity algorithm is used to devise the optimal frequency-division duplex plan for very-high-speed digital subscriber lines.**

***Index Terms*—Bit loading, convex optimization, digital subscriber line (DSL), discrete multitone (DMT), frequency division duplex (FDD), frequency-division multiple access, frequency-selective channel, intersymbol interference, multiaccess communication, orthogonal frequency division multiplexing (OFDM).**

I. Introduction

In a Gaussian multiple-access channel, M independent senders simultaneously communicate with a single receiver in the presence of additive Gaussian noise. The Shannon capacity region of the multiple-access channel refers to the set of simultaneous achievable rates $(R_1, R_2, \ldots, R_M)$ at which the receiver may decode information from each sender without error. For example, the following is a two-user memoryless Gaussian multiple-access channel

$$Y = X_1 + X_2 + Z \tag{1}$$

where X_1 and X_2 are the input signals under power constraints P_1 and P_2, respectively, Y is the output signal, and Z is the additive white Gaussian noise (AWGN) with a noise power-spectral-density N. The capacity region of this multiple-access channel is a pentagon [1]:

$$R_1 \le W \log\left(1 + \frac{P_1}{WN}\right) \tag{2}$$

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The authors are with the Electrical Engineering Department, Stanford University, Stanford, CA 94305-9515 USA (e–mail: weiyu@dsl.stanford.edu; cioffi@stanford.edu).

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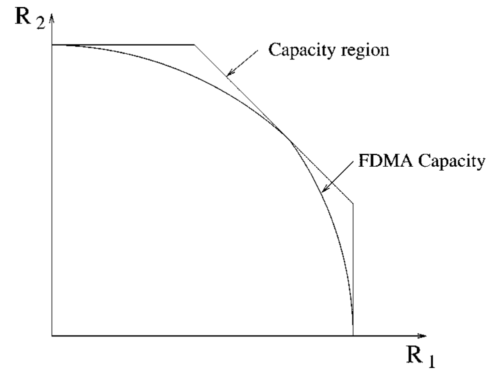

Fig. 1. The Shannon capacity region and the FDMA-capacity region of a multiple-access channel.

$$R_2 \le W \log\left(1 + \frac{P_2}{WN}\right) \tag{3}$$

$$R_1 + R_2 \le W \log\left(1 + \frac{P_1 + P_2}{WN}\right) \tag{4}$$

where W is the total available bandwidth. To achieve the capacity for a multiple-access channel, joint decoding at the receiver is needed in general. For the memoryless channel example above, it is well known that the corner points of the capacity region are achieved with both users transmitting at the same time and in the same frequency band. The combined signals are then separated at the receiver using a successive decoding technique, i.e., one user is decoded first, then its effect is subtracted before the second user is decoded. Note that the optimal transmission strategy for the multiple-access channel requires the entire frequency band to be used by both users simultaneously and frequency-division multiple access (FDMA) is not optimal except in special cases. In fact, the capacity region corresponding to the FDMA strategy for a memoryless multiple-access channel is

$$R_1 \le \alpha W \log\left(1 + \frac{P_1}{\alpha WN}\right) \tag{5}$$

$$R_2 \le (1-\alpha) W \log\left(1 + \frac{P_2}{(1-\alpha)WN}\right) \tag{6}$$

where α is the proportion of the total bandwidth used by the first user. Fig. 1 shows both the rate region achievable with FDMA and the Shannon capacity region. It is clear that the FDMA capacity region is strictly smaller than the Shannon capacity region and FDMA is optimal only at a single point [2]. Incidentally, this point corresponds to an FDMA strategy where each user's share of bandwidth is proportional to its respective power. The tangent line at this point is at 45 degrees and it corresponds to the maximum sum capacity point.

These ideas can be generalized to the Gaussian multiple-access channel with intersymbol interference (ISI). The capacity region in this case was characterized by Cheng and Verdu [3]. The idea is to decompose the channel in the frequency domain and to divide the channel into parallel independent memoryless subchannels along the frequency dimension. For any given power allocation over the subchannels, the achievable rate region is again a pentagon. The capacity region of the multiple-access channel is then the union of the pentagons over all possible power allocations. The union is not necessarily a pentagon and, because each user has a different channel, finding the optimal allocation of power over the frequencies is not a trivial task. The optimal power allocation is different for different points in the capacity region and it can only be found numerically [3]. In general, the optimal spectra for the two users overlap in frequency so again frequency-division multiple access is not optimal except in special cases. It turns out that, as for the non-ISI channels, the special point in the capacity region where FDMA is optimal again corresponds to the rate-sum maximization point.

Although not optimal in the information theoretical sense, the frequency-division multiple-access technique is often desirable from a practical implementation point of view. An FDMA transmission scheme allows different users to occupy orthogonal dimensions, so they can be separated at the receiver without joint decoding. This greatly simplifies the receiver design and it is especially suitable in the orthogonal frequency division multiplex (OFDM) systems. For this reason, this paper will concentrate on the frequency-division multiple-access technique. A system using FDMA needs to assign different frequency bands to different users. However, when the channels have ISI and when each user's channel is different, finding the optimal allocation of frequency among the different users is in general not easy. This paper will focus on this problem and the objective is to numerically characterize the capacity region for the ISI multiple-access channel with the FDMA restriction.

The study of FDMA-capacity region is motivated by the following problem in the design of very-high-speed digital subscriber line (VDSL) systems. The VDSL system uses an ordinary telephone twisted-pair to transmit high-speed data between the central office and the customer premise. The twisted-pair is a severe ISI channel. The upstream and the downstream transmissions in VDSL are separated by a frequency-division duplex (FDD) scheme. The VDSL system studied in this paper uses a modulation scheme based on discrete multitone (DMT) or OFDM which allows an arbitrary frequency division between the upstream and the downstream. In an FDD design, it is natural to optimize the partition of the upstream and the downstream transmissions. It turns out that finding the optimal frequency partition between the upstream and the downstream is exactly the same problem as finding the optimal FDMA-capacity region for a multiple-access channel with two users.

The idea of using FDMA as a possible joint signaling strategy for two DSL modems has appeared in a related work [4], where the problem of avoiding excessive near-end crosstalk was studied. The authors concluded that FDMA should be used when the crosstalk level is high, although the problem of optimal frequency partition was not approached from a power-constrained capacity viewpoint. The FDMA-capacity problem also appears in the wireless OFDM context. For example, Wong *et al.* [5] considered the problem of finding the optimal frequency partition to minimize the total transmission power in a mulitple-access OFDM system while satisfying a minimum rate constraint for each user. This power minimization problem is essentially the dual of the capacity problem considered in this paper. So, the techniques proposed in [5] are applicable here as well. In particular, [5] formulated the problem as an integer programming problem and used the idea of continuous relaxation to approximate the solution. This same approach will be used here also. In another related work, Hoo *et al.* [6], [7] considered the problem of finding the optimal frequency partition to support several data streams with different qualities of service (QoS) in an OFDM system. In [6] and [7], several different problem formulations were proposed along with a low-complexity suboptimal algorithm to find the optimal solution. Although closely related, the problem formulations in [6] and [7] have a common total power constraint for all data streams, while the problem considered in this paper has a separate power constraint for each user. Nevertheless, the low-complexity algorithm proposed in [6] and [7] turns out to be applicable in the present context. In fact, one of the main results of this paper is to characterize conditions under which the low-complexity algorithm applies.

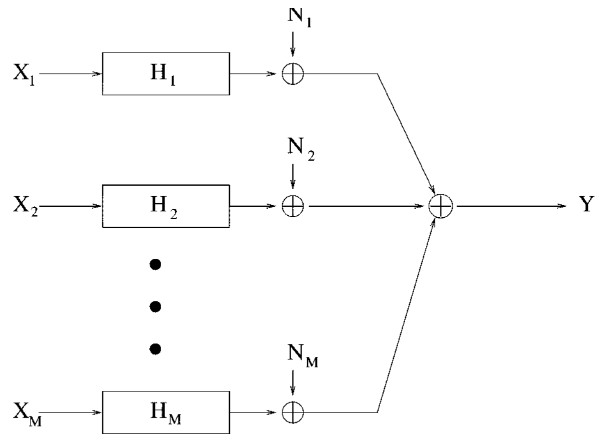

Fig. 2. A mulitple-access channel.

The rest of the paper is organized as follows. The FDMA-capacity problem for the mulitple-access channel is formulated in Section II. The problem is then shown to have a convex relaxation, thus allowing general convex programming algorithms to be used to solve the problem numerically. Section III shows that under two special cases the optimal two-user FDMA partition is a two-band partition and proposes a low-complexity suboptimal solution based on this result. Section IV describes the VDSL duplex problem as a practical example and presents numerical simulation results. Conclusions are drawn in Section V.

II. Optimal Frequency Partition

A Gaussian mulitple-access channel with M users is shown in Fig. 2, where H_i and N_i denote the channel response and the noise process for the ith user, respectively. The objective is to characterize the capacity region under the restriction that the power spectra for different users are nonoverlapping in the frequency domain. Clearly, the FDMA-capacity region is a proper subset of the capacity region without the FDMA constraint.

The capacity region is a concept that is used to capture the tradeoff among the individual data rates for the different users in a multiuser communication situation. The capacity region is defined to be the set of rates $(R_1, \ldots, R_M)$ simultaneously achievable for all users. For the Gaussian mulitple-access channel, the capacity region is convex. The tradeoff exists because system resources are limited and different users compete for the limited resources. The tradeoff is best characterized by solving a set of optimization problems parameterized by a relative priority for each user, $(\alpha_1, \alpha_2, \ldots, \alpha_M)$, where $\alpha_i \geq 0$. More specifically, the aggregate data rate

$$R = \sum_{i=1}^{M} \alpha_i R_i \tag{7}$$

is maximized at a boundary point of the achievable rate region, where the normal vector to the tangent hyperplane at this point is $(\alpha_1, \alpha_2, \ldots, \alpha_M)$. Maximizing this aggregate data rate for all possible α_i's traces out the entire rate region. It is important to recognize that solving for a single point in the rate region, such as the maximum rate-sum point, is not sufficient. Consider the case where one user has a much better channel than do all other users. Maximizing the sum rate typically results in a rate combination where the user with the best channel will have a much higher rate than all other users. This may not be the desirable operating point in a system design.

Mathematically, let $H_i(\omega)$ and $N_i(\omega)$ be the channel transfer function and the noise power-spectral-density for the ith user in a Gaussian mulitple-access channel. The goal is to find the optimal transmit power-spectral density $P_i(\omega)$ for each user that collectively maximizes some weighted sum rate subject to the FDMA constraint. Using the Shannon capacity formula for the Gaussian channel, the optimization problem can be expressed as follows:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{M} \alpha_i \int_0^W \log\left(1 + \frac{P_i(\omega)\,|H_i(\omega)|^2}{N_i(\omega)}\right) d\omega \\ \text{subject to} \quad & \int_0^W P_i(\omega) d\omega \leq \mathbf{P}_i \quad \forall i \\ & P_i(\omega) \geq 0 \quad \forall i \\ & P_i(\omega) P_j(\omega) = 0 \quad \forall i, j, i \neq j \end{aligned} \tag{8}$$

where $\mathbf{P}_i$ is the power constraint for user i. The constraint $P_i(\omega)P_j(\omega) = 0$ guarantees that the power spectra for the different users do not overlap in frequency and it must be satisfied for all i and j, $i \neq j$. The key to solving the above problem is to find the optimal partition of the frequency spectrum among the different users. Once the frequency band assignment is fixed, the optimal spectrum for each user is just the water-filling spectrum within the assigned band. Intuitively, several factors need to be considered in the frequency band assignment. First, the user with a higher priority, α_i, should be favored because its rate has a higher weight in the objective. Secondly, each frequency should be assigned to the user who can make the best use of it. So, the user with a better channel-gain-to-noise ratio, $|H_i(\omega)|^2/N_i(\omega)$, should be favored at the frequency ω. Thirdly, the user with a higher total power constraint should be assigned frequencies more generously because it has more power available and is able to use frequencies more efficiently. So, in deciding the optimal frequency partition, it is necessary to strike a balance among how good each channel is, how much total power each user has, and the relative priorities among the users. An example of such a compromise is captured in the "equivalent channel" idea in [3]. In deriving the optimal FDMA spectrum for the special case of maximizing the sum rate, Cheng and Verdu showed that a balance can be found by a proper scaling of the channel and the power constraint. In the following, we will devise similar numerical algorithms for the general case.

The first idea is to discretize the problem by dividing the frequency spectrum into a large number of frequency bins of finite width each. This allows a formulation of the problem in a finite dimensional space. The channel frequency response and the noise power-spectral-density are assumed to be flat within each bin and as the number of bins increases to infinity, this piece-wise constant channel model converges to the actual channel. In this discretized version, the frequency partition problem is reduced to a frequency-bin assignment problem. In effect, the frequency partition boundaries between the two users are now restricted to the bin boundaries. This is equivalent to introducing new variables $\omega_{i,j}$, where

$$\omega_{i,j} = 0 \text{ or } W_j \tag{9}$$

indicating whether or not the jth frequency bin is used by the ith user. Here, W_j is the width of the frequency bin j. The FDMA constraint then becomes:

$$\omega_{m,j}\omega_{n,j} = 0 \qquad \forall m, n, m \neq n \tag{10}$$

which ensures that each frequency bin is used by one user only. Unfortunately, this bin-assignment problem belongs to the class of integer programming problems, for which an exact solution usually involves an exhaustive search. An exhaustive search is computationally prohibitive when the number of frequency bins is large, so an efficient way to solve this problem is called for.

One approach to the integer programming problem is to approximate the problem by its continuous relaxation, which is hopefully easier to solve. The difficulty in solving an integer programming problem lies in the fact that the constraint set is a set of isolated points. The idea of continuous relaxation is to enlarge the constraint set to include all convex combinations of the original points. A convex constraint set can be dealt with much more easily. This approach was taken in an earlier work [5] for a similar problem. This trick is the following. Instead of forcing the optimization variable to be either 0 or W_j, constraint (9) can be relaxed to:

$$0 \leq \omega_{i,j} \leq W_j. \tag{11}$$

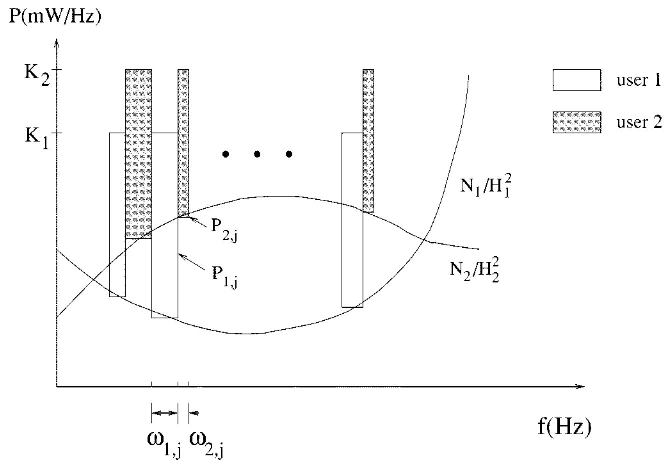

Fig. 3. Optimization with frequency-division multiple access.

Likewise, constraint (10) may be relaxed to

$$\sum_{i=1}^{M} \omega_{i,j} \leq W_j. \tag{12}$$

It is easy to check that the new constraints define a set of points that are precisely the convex combination of all points satisfying the original constraints. In fact, instead of restricting the boundaries of frequency partition to align with the bin boundaries, the boundaries are now allowed to be anywhere within the bin, hence relaxing the integer programming problem into a constrained continuous-variable optimization problem. As illustrated in Fig. 3, each frequency bin can now be sub-partitioned arbitrarily between the two users. Since the sub-partition is done in the frequency domain, this approach can also be thought of as an exact formulation of the FDMA-capacity problem for the case where the channel frequency responses and the noise power-spectral-densities are piece-wise constant. Because piece-wise constant functions can approximate a continuous function arbitrarily well, the solution to the continuous relaxation is expected to be close to the optimum as the bin width becomes small. However, the continuous relaxation does not necessarily yield a solution where $\omega_{i,j}$ is either 0 or W_j. If such integer solution is required, it is necessary to round the possibly fractional values to 0 or W_j. But this is rarely a problem if the number of bins is large compared to the number of users. In fact, as the bin width becomes small, at the optimum, almost every bin except a few will be exclusively assigned to one user. This is because the optimal frequency allocation usually involves only a few frequency bands. When the width of the frequency bin is less than the width of the narrowest frequency band in the optimal allocation, frequency bins in the middle of the bands are fully allocated to one user only and those on the boundary of the frequency bands are the only ones shared. As long as the number of such shared bins is small, the boundary bins may be assigned to either user arbitrarily without affecting the total rate appreciably. An integer solution is often required in practice because frequency-division multiple access can be naturally implemented using OFDM where an IFFT/FFT pair together with a cyclic prefix are used to perform the modulation and de-modulation functions. Each OFDM tone corresponds to a frequency bin. In this case, because the FFT-size is fixed in advance, no further subdivision of each tone is possible in the frequency domain. However, it is possible to subdivide the tones in the time domain if the users can be synchronized. This interpretation was given in [5].

Mathematically, the continuous relaxation of the bandwidth assignment problem can be posed as follows. The entire frequency band is divided into N bins. Let $\omega_{i,j}$ and $P_{i,j}$ be respectively the bandwidth and power assigned to the user i in the frequency bin j. The objective is to choose $\omega_{i,j}$ and $P_{i,j}$ to maximize the aggregate data rate:

$$\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{M} \alpha_i \sum_{j=1}^{N} \omega_{i,j} \log\left(1 + \frac{P_{i,j}\,|H_{i,j}|^2}{\omega_{i,j} N_{i,j}}\right) \\
\text{subject to} \quad & \sum_{i=1}^{M} \omega_{i,j} \leq W_j \qquad \forall j \\
& \sum_{j=1}^{N} P_{i,j} \leq \mathbf{P}_i \qquad \forall i \\
& P_{i,j} \geq 0 \qquad \forall i,j \\
& \omega_{i,j} \geq 0 \qquad \forall i,j
\end{aligned} \tag{13}$$

where M is the total number of users, α_i's are the relative priorities for each user, $H_{i,j}$ and $N_{i,j}$ are the channel frequency response and the noise power-spectral-density for the user i in the frequency bin j, W_j is the width of the jth frequency bin and $\mathbf{P}_i$ is the power constraint for the user i. If a power-spectral-density limit is also needed, the following additional constraint can be added:

$$\frac{P_{i,j}}{\omega_{i,j}} \leq S_{i,j} \tag{14}$$

where $S_{i,j}$ is the maximum power-spectral-density for the user i in the frequency bin j. This constrained optimization problem is considerably easier to solve than the integer programming problem. The key observation is that the objective function is concave.

Lemma 1: The objective function in the optimization problem above is a concave function in $(P_{i,j}, \omega_{i,j})$.

Proof: Observe that the objective function (13) is a positive linear combination of functions of the type $f(x,y) = x\log(1+y/x)$, where $x \geq 0$ and $y \geq 0$. Since a positive linear combination of concave functions is concave, to prove the concavity of the objective, it is only necessary to show that $f(x,y)$ is concave in (x,y) in the first quadrant.

A two-dimensional function is concave if and only if its restriction to any line is concave [8]. Let $g(x) = f(x,y)|_{y=ax+b}$, then

$$g(x) = x\log\left(1+\frac{y}{x}\right)\Big|_{y=ax+b} = x\log\left(1+a+\frac{b}{x}\right). \tag{15}$$

$g(x)$ is concave for $x > 0$ and this can be verified by taking its second derivative:

$$g'' = \frac{\dfrac{b}{1+a}}{x+\dfrac{b}{1+a}}\left(\frac{1}{x+\dfrac{b}{1+a}} - \frac{1}{x}\right). \tag{16}$$

Consider three cases:

- $b/(1+a) \geq 0$: Since x is nonnegative, the first term in $g''(x)$ is nonnegative. The second term is negative. So, the product is negative.
- $-x < b/(1+a) < 0$: The numerator of the first term is negative and the denominator is positive, so the first term in $g''(x)$ is negative. The second term is positive because x is positive and $b/(1+a)$ is negative. So, the product is negative.
- $b/(1+a) \leq -x$: In this case, both the numerator and the denominator of the first term is negative, so the first term is positive. The second term is negative since both fractions are negative. Again, the product is negative.

The second derivative is always negative. So, $g(x)$ is concave when $x \geq 0$, $f(x, y)$ is concave in the first quadrant and (13) is concave in $(P_{i,j}, \omega_{i,j})$. □

Now, observe also that the constraints in the optimization problem are linear. So the constraint set is convex and the optimization problem takes the form of maximizing a concave function subject to a convex constraint. This is the standard form of a convex programming problem. In convex programming problems, a local maximum is also a global maximum, so numerical search algorithms such as the interior-point method are well suited to obtain solutions efficiently [8]. In fact, many standard software packages are available and the complexity of numerical methods increases as a polynomial function of the problem size. For this reason, once the problem is transformed into a convex problem, it can be considered numerically tractable.

III. A Low Complexity Algorithm

Although convex programming problems are numerically stable and they can be solved much more efficiently than integer programming problems, its computational complexity still depends on the number of optimizing variables, which in this case can be large if the number of frequency bins is large. General-purpose convex programming algorithms take advantage of the convexity of the problem, but it does not otherwise explore the specific problem structure. Exploring the problem structure can lead to intuitions on the structure of the solution that are otherwise lacking in a purely numerical approach. Such intuition can lead to further reduction in problem dimensionality and run-time complexity that are important if spectrum allocation is performed dynamically. The idea is to search through a subset of frequency partitions and hope that the optimum in the subset is close to the global optimum. Such an approach was previously taken in [6] and [7], where the problem of finding the optimal frequency partition to guarantee different qualities of service (QoS) for multiple data streams in a single subscriber line is considered. The search algorithm proposed in [6] and [7] assigns the better subchannels to the user with a higher priority, and this was shown to be asymptotically optimal. Although the FDMA multiple-access channel capacity problem considered in this paper differs from those in [6] and [7], the low-complexity algorithm proposed in [6] and [7] turns out to be applicablle here as well under certain conditions. The aim of this section is to characterize these conditions precisely and to derive similar algorithms for the general case.

To gain some intuition, consider a two-user case where the two users have identical channel transfer functions and noise power-spectral-densities. First, consider the case where both users have the same priority, i.e., the objective is to maximize $R_1 + R_2$. Given a frequency partition, the optimal power allocation within each user is just the water-filling allocation. It turns out that in the special case where the channels and the user priorities are the same, the optimal FDMA partition is a partition that results in the same water-filling level for both users [3]. This water-filling level can be found directly by water-filling the common channel with the combined power. In fact, the optimal frequency partition is not unique in this case. For example, each frequency bin may be divided into two halves in proportion to the power constraints, or the sub-channels may be divided into two contiguous bands. As long as the frequency band boundary is chosen so that the two users have the same water-filling level, the same maximum sum data rate is achieved regardless of which user is assigned the better sub-channels.

The intuitive reason behind the above argument is that when the two users have the same priority, the exact assignment for each sub-channel is not important so long as each sub-channel is fully utilized. This is not the case when the two users have different priorities. In this situation, the two users do not have the same water-filling level at the optimum and the sub-channels cannot be assigned arbitrarily. Nevertheless, if the two users' channel characteristics are the same, the optimal frequency partition turns out to be a simple two-band partition. The optimal frequency assignment always assigns the better sub-channels to the user with higher priority. This is stated in the following result.

Theorem 1: Consider a two-user Gaussian mulitple-access channel. Define the channel-gain-to-noise ratio as $g_{i,j} = |H_{i,j}|^2/N_{i,j}$, *where* $i = 1, 2$ *and* $j = 0, \ldots, N$. *Assume that the channel-gain-to-noise ratios for the two users are the same, i.e.,* $g_{1,j} = g_{2,j} = g_j, \forall j$. *Without loss of generality, let* g_j *be monotonically decreasing, i.e.,* $g_m \geq g_n$ *for* $m < n$. *Then, the optimal frequency partition maximizing* $\alpha_1 R_1 + \alpha_2 R_2$, *where* $\alpha_1 < \alpha_2$, *consists of two frequency bands only. More precisely, at the optimum, there exist* L_1 *and* L_2, $1 \leq L_1 \leq L_2 \leq N$, *such that* $\omega_{1,j} = 0$ *and* $\omega_{2,j} = W_j$ *for all* $j < L_1$ *and* $\omega_{1,j} = W_j$ *and* $\omega_{2,j} = 0$ *for all* $L_1 < j < L_2$. *Frequency bins beyond* L_2 *are not used by either user. Only the variables* ω_{1,L_1} *and* ω_{2,L_1} *may take values between 0 and* W_j.

This result is based on the Karush–Kuhn–Tucker (KKT) condition for the optimization problem (13) with $M = 2$. Assume a general channel model where $g_{1,j}$ is not necessarily equal to $g_{2,j}$ for now. Form the Lagrangian as follows:

$$\begin{aligned} L(\omega_{i,j}, P_{i,j}) = & \sum_{i=1}^{2} \alpha_i \sum_{j=1}^{N} \omega_{i,j} \log\left(1 + \frac{P_{i,j}\,|H_{i,j}|^2}{\omega_{i,j} N_{i,j}}\right) \\ & + \sum_{j=1}^{N} \lambda_j \left(\sum_{i=1}^{2} \omega_{i,j} - W_j\right) \\ & + \sum_{i=1}^{2} \mu_i \left(\sum_{j=1}^{N} P_{i,j} - \mathbf{P}_i\right) - \sum_{i=1}^{2}\sum_{j=1}^{N} \nu_{i,j} P_{i,j} \\ & - \sum_{i=1}^{2}\sum_{j=1}^{N} \kappa_{i,j}\omega_{i,j}, \end{aligned} \tag{17}$$

where λ_j, μ_i, $\nu_{i,j}$ and $\kappa_{i,j}$ are Lagrange multipliers which take on positive values. The KKT condition is derived by taking the derivatives of the Lagrangian with respect to $\omega_{i,j}$ and $P_{i,j}$ and setting them to zero. Taking the derivative with respect to $P_{i,j}$ gives the KKT condition corresponding to the usual water-filling condition: there exist positive constants K_i, such that for all $i = 1, 2$ and for all $j = 1, \ldots, N$, if $P_{i,j} > 0$, then

$$\frac{P_{i,j}}{\omega_{i,j}} + \frac{1}{g_{i,j}} = K_i, \tag{18}$$

and if $P_{i,j} = 0$, then

$$\frac{1}{g_{i,j}} \geq K_i. \tag{19}$$

Taking the derivative with respect to $\omega_{i,j}$ gives the second KKT condition: for all $j = 1 \ldots N$, if $\omega_{1,j} > 0$ and $\omega_{2,j} > 0$, then

$$\alpha_1 \log\left(1 + \frac{P_{1,j}g_{1,j}}{\omega_{1,j}}\right) - \alpha_1 \frac{\frac{P_{1,j}g_{1,j}}{\omega_{1,j}}}{1 + \frac{P_{1,j}g_{1,j}}{\omega_{1,j}}} = \alpha_2 \log\left(1 + \frac{P_{2,j}g_{2,j}}{\omega_{2,j}}\right) - \alpha_2 \frac{\frac{P_{2,j}g_{2,j}}{\omega_{2,j}}}{1 + \frac{P_{2,j}g_{2,j}}{\omega_{2,j}}}. \tag{20}$$

The left-hand side of above equation can be interpreted as the marginal benefit of extra bandwidth for user 1 in the frequency bin j. The right-hand side is the marginal benefit of extra bandwidth for user 2 in the frequency bin j. If a frequency bin is shared between the two users, the marginal benefits for the two users should be equal. If a frequency bin is exclusively used by user 2, i.e., $\omega_{1,j} = 0$ and $\omega_{2,j} = W_j$, then the left-hand side should be strictly less than the right-hand side. Likewise, if a frequency bin is used exclusively for user 1, i.e., $\omega_{1,j} = W_j$ and $\omega_{2,j} = 0$, then the inequality should be reversed[1]. Because the problem is concave, (18) and (20), together with the total power constraints $\sum_{j=1}^{N} P_{i,j} = \mathbf{P}_i$, total bandwidth constraints $\sum_{i=1}^{2} \omega_{i,j} = W_j$ and the positivity constraints on $P_{i,j}$ and $\omega_{i,j}$ are the necessary and sufficient optimality conditions.

Equation (18) is the condition for the optimal power allocation within each user. Fixing the frequency partition, the condition is just the classical single-user water-filling condition among the sub-channels in use. Equation (20) is the condition for the optimal bandwidth allocation between the two users while keeping the power allocation fixed. Equations (18) and (20) together achieve a balance between optimal power allocation and optimal bandwidth allocation. Equations (18) and (20) are nonlinear and there is no analytic solution in general. However, these conditions allow us to characterize the structure of the solution. We are now ready to prove theorem 1.

Proof of Theorem 1: Let K_1, K_2 be user 1 and user 2's respective water-filling levels at the optimal power and bandwidth allocation. The objective is to prove that there exist L_1 and L_2 such that the left-hand side of (20) is strictly less than the right-hand side for $1 \leq j < L_1$, the left-hand side of (20) is strictly greater than the right-hand side for $L_1 < j < L_2$ and beyond L_2, $1/g_j \geq K_1$ and $1/g_j \geq K_2$. This would imply that user 2 occupies the frequency bins below L_1, user 1 occupies the frequency bins between L_1 and L_2 and neither user uses bins beyond L_2.

If a frequency bin is shared between the two users, both (18) and (20) need to be satisfied with equality. Substituting (18) into (20) gives:

$$\alpha_1 \log(K_1 g_j) + \frac{\alpha_1}{K_1 g_j} - \alpha_1 = \alpha_2 \log(K_2 g_j) + \frac{\alpha_2}{K_2 g_j} - \alpha_2. \tag{21}$$

This equality is satisfied in each shared bin. If a bin is assigned exclusively to user 2, the left-hand side of (21) is strictly less than the right-hand side. If a bin is assigned exclusively to user 1, the left-hand side of (21) is strictly greater. Consider the difference between the left-hand side and right-hand side as a function of $1/g_j$, call it $f(x)$,

$$f\left(\frac{1}{g_j}\right) = (\alpha_2 - \alpha_1) \log\left(\frac{1}{g_j}\right) + \left(\frac{\alpha_1}{K_1} - \frac{\alpha_2}{K_2}\right)\left(\frac{1}{g_j}\right) + \alpha_1 \log K_1 - \alpha_2 \log K_2 - \alpha_1 + \alpha_2. \tag{22}$$

When $f(1/g_j) < 0$, the jth frequency bin is assigned to user 2. When $f(1/g_j) > 0$, the jth frequency bin is assigned to user 1. Therefore, when g_j's are sorted, every time $f(x)$ crosses zero, the frequency bin assignment switches from one user to the other.

Consider user 2 first. We will show that the frequency assignment for user 2 is a single band. To prove this, it is only necessary to show that $f(x)$ has at most one root in the range $0 < x < K_2$. The upper range is K_2 because user 2 only uses frequency bins where $1/g_n$ is less than its water-filling level K_2.

Let $\alpha_1 < \alpha_2$. Consider two cases:

- $K_2/K_1 \geq \alpha_2/\alpha_1$: In this case, $\alpha_1/K_1 - \alpha/K_2 \geq 0$ and $\alpha_2 - \alpha_1 > 0$. Both $\log(x)$ and x are increasing functions of x, so $f(x)$ is increasing and can have only one root.
- $K_2/K_1 < \alpha_2/\alpha_1$: In this case, $\alpha_1/K_1 - \alpha_2/K_2 \leq 0$. By differentiating $f(x)$, it is easy to verify that $f(x)$ is strictly increasing until it reaches a maximum after which point it becomes strictly decreasing. So, $f(x)$ can potentially have two roots, one in the increasing segment and the other in the decreasing segment. However it is easy to check that $f(K_2) = \alpha_1(K_2/K_1 - \log(K_2/K_1) - 1) \geq 0$. So $f(x)$ cannot have decreased to zero at $x = K_2$. Therefore, in the range of interest $x < K_2$, $f(x)$ can only have one root.

In both cases, $f(x)$ crosses zero only once. Since g_j is sorted from the largest to the smallest, there exists L_1 such that $f(1/g_j) < 0$ for $j < L_1$ and $f(1/g_j) > 0$ for $j > L_1$. So, the frequency bins $1 \leq j < L_1$ are used exclusively by user 2, the frequency bin L_1 is shared by both users and the frequency bins $L_1 < j \leq N$ are not used by user 2 at all. To determine the set of frequency bins used by user 1, water-filling can then be performed on frequency bins $L_1 < j \leq N$. Since g_n are sorted, user 1 will use bins from $L_1 + 1$ to some $L_2 \leq N$. Bins beyond L_2 are not used by either user. □

[1]There is a singularity in the KKT condition when $\omega_{i,j} = 0$. The KKT condition should be interpreted in the limiting sense. Note that $\omega_{i,j} = 0$ also implies that $P_{i,j} = 0$. A rigorous derivation of the KKT condition can be obtained by replacing the constraint $\omega_{n,m} \geq 0$ with $\omega_{n,m} \geq \epsilon$, then letting ϵ go to zero.

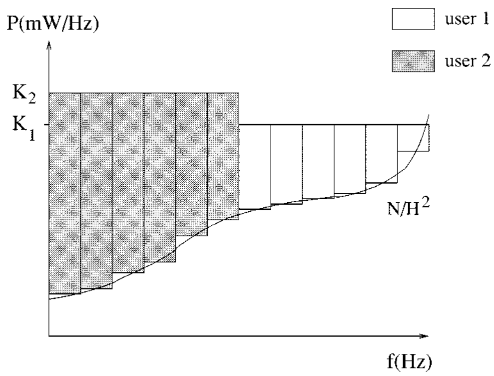

Fig. 4. A typical two-band frequency assignment.

Fig. 4 illustrates a typical frequency assignment where the better sub-channels are assigned to the user with higher priority. When the conditions of Theorem 1 are satisfied, all frequency bins are exclusively assigned to either user 1 or user 2, except the single boundary bin. When the number of frequency bins is large, the data rate contributed by one frequency bin is small. The boundary bin may be arbitrarily assigned to either user 1 or user 2 to obtain a discrete assignment that is close to the optimum. Theorem 1 suggests that to find the optimal discrete frequency partition, for the case where the two users' channels are the same, it is only necessary to search through all two-band partitions. This is a considerable saving in computational complexity compared to the general-purpose convex optimization approach. In fact, the condition of Theorem 1 can be somewhat relaxed. As the following theorem shows, when the signal-to-noise ratio is high, even if the two users' channels are not the same, the two-band partition remains optimal.

Theorem 2: Consider a two-user Gaussian mulitple-access channel. Without loss of generality, assume that $g_{1,j}^{\alpha_1}/g_{2,j}^{\alpha_2}$ *is decreasing in* j. *If at the optimum frequency partition, the signal-to-noise ratio, defined as* $SNR_{i,j} = K_i g_{i,j} - 1$, *is much larger than 1 in every frequency bin and for each user, then the optimal frequency partition that maximizes* $\alpha_1 R_1 + \alpha_2 R_2$ *consists of two contiguous frequency bands with user 1 using the lower frequency bins and user 2 using the higher frequency bins.*

Proof: At the optimum frequency partition, if $\omega_{i,j} > 0$ for the user i in the frequency bin j, the definition of SNR in the theorem reduces to the conventional definition: $SNR_{i,j} = K_i g_{i,j} - 1 = P_{i,j} g_{i,j}/\omega_{i,j}$. Now, for the frequency bin j to be shared between the two users, (20) needs to be satisfied with equality. Substituting the SNR definition into (20), we get

$$\alpha_1 \log(1 + SNR_{1,j}) - \alpha_1 \frac{SNR_{1,j}}{1 + SNR_{1,j}} = \alpha_2 \log(1 + SNR_{2,j}) - \alpha_2 \frac{SNR_{2,j}}{1 + SNR_{2,j}}. \quad (23)$$

When a frequency bin is used exclusively by one user, the above equality becomes inequality. If the left-hand side is greater than the right-hand side, then user 1 should use the frequency bin j and vice versa for user 2.

At high SNR, the fraction $SNR_{i,j}/(1 + SNR_{i,j})$ on either side of (23) can be approximated by 1. Now, let K_1 and K_2 be the respective water-filling levels for the two users at the optimum, substitute (18) into (23), take the difference between the left-hand side and the right-hand side of (23) and call the function $f(g_{1,j}^{\alpha_1}/g_{2,j}^{\alpha_2})$. To decide whether a frequency bin is used by user 1 or user 2, it is only necessary to decide whether the following function is greater than zero or less than zero:

$$f\left(\frac{g_{1,j}^{\alpha_1}}{g_{2,j}^{\alpha_2}}\right) \approx \alpha_1 \log(K_1 g_{1,j}) - \alpha_2 \log(K_2 g_{2,j}) + \alpha_2 - \alpha_1 = \log\left(\frac{g_{1,j}^{\alpha_1}}{g_{2,j}^{\alpha_2}}\right) + \log\left(\frac{K_1^{\alpha_1}}{K_2^{\alpha_2}}\right) + \alpha_2 - \alpha_1. \quad (24)$$

Now, since $g_{1,j}^{\alpha_1}/g_{2,j}^{\alpha_2}$ is decreasing in j and the logarithm is an increasing function, $f(g_{1,j}^{\alpha_1}/g_{2,j}^{\alpha_2})$ is a decreasing function in j. Therefore, there exists L_1 such that $f(g_{1,j}, g_{2,j}) > 0$ for $j < L_1$ and $f(g_{1,j}, g_{2,j}) < 0$ for $j > L_1$. So the frequency bins below L_1 are exclusively used by user 1. Following the same argument as in the proof of Theorem 1, this implies that the optimum partition is a two-band solution. □

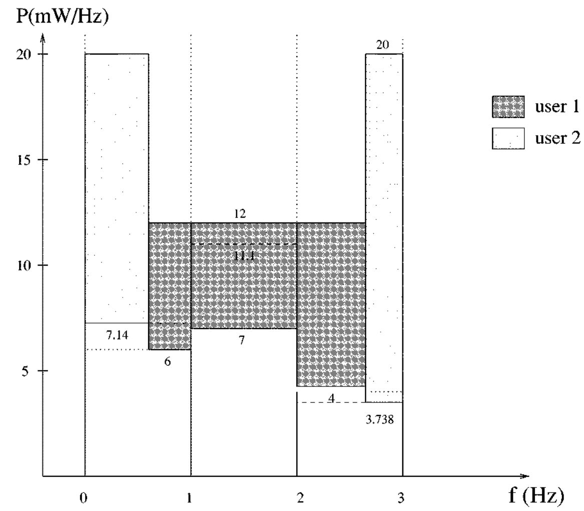

Fig. 5. A multiband optimum.

Assuming the SNR being much larger than 1 is equivalent to assuming that the spectral efficiency is much larger than 1 bit/Hz in all frequencies. The intuition is that at a high SNR, the structure of the optimum frequency partition does not depend on the power constraints and it is solely a function of the channel characteristics. This is not true at a low SNR. The following numerical example illustrates a two-user three-frequency-bin case where a multiple-band solution is the optimum. Here, the bandwidth for each frequency bin has a width of 1 Hz. The power constraints for the two users are 2 mW and 4.7 mW respectively. The channel gain is 1 in all bins. The noise power-spectral-density in mW/Hz for the two users are (6, 7.145) in bin 1, (7, 11.1) in bin 2 and (4, 3.738) in bin 3 respectively. The optimal frequency partition and power allocation that maximize $2R_1 + R_2$ are illustrated in Fig. 5. Note that the frequency bins are monotonic in $g_{1,j}^{\alpha_1}/g_{2,j}^{\alpha_2}$, but both frequency bin 1 and bin 3 are shared between the two users. The SNR here is too low for Theorem 2 to hold.

Theorems 1 and 2 suggest the following low-complexity approach that finds a near-optimal frequency partition by searching through a set of two-band partitions.

Algorithm 1 A low-complexity algorithm for finding a near-optimal frequency partition that maximizes $\alpha_1 R_1 + \alpha_2 R_2$:

1. *Sort the sub-channels according to* $g_{1,j}^{\alpha_1}/g_{2,j}^{\alpha_2}$ *from the largest to the smallest. In case of a tie, sort further according to* $g_{1,j}$.
2. *For each* $j = 0, \ldots, N$,
   - *water-fill for user 1 using frequency bins 1 to* j,
   - *water-fill for user 2 using frequency bins* $j+1$ *to* N,
   - *compute* $\alpha_1 R_1 + \alpha_2 R_2$.
3. *Choose the frequency partition boundary to be the one that maximizes* $\alpha_1 R_1 + \alpha_2 R_2$.

The algorithm can be further improved using a binary search on the boundary bin. Binary search is feasible because for each bin j, (20) can be used to decide whether the optimal boundary is larger than or smaller than j. Using a binary search, the run-time complexity of the algorithm is $O(N \log N)$ because sorting takes $O(N \log N)$ operations, each water-filling takes $O(N)$ operations on the sorted channels and there are at most $O(\log N)$ water-fillings to do. In practice, this sub-optimal algorithm is much faster than general-purpose convex programming algorithms. A special case of this algorithm ($\alpha_1 = \alpha_2$) was noted in [9]. Theorem 1 and Theorem 2 guarantee that the algorithm will find a near-optimal solution either when the two channels are the same, or when the SNR is high. However, as the example in the next section illustrates, in the context of finding the optimal duplex scheme for VDSL, this low-complexity algorithm works well even when these assumptions do not hold.

IV. Optimal Duplex in VDSL

The algorithm proposed in the previous section is now used to solve the optimal frequency duplex problem in the Very-high-speed Digital Subscriber Line (VDSL) systems. VDSL is designed to carry high speed digital data over copper-based twisted-pairs by utilizing the frequency spectrum up to 20 MHz. While functional requirements for VDSL are still being standardized, it is expected that symmetric data rates up to 26 Mbps can be achieved on short lines (1000 ft), 13 Mbps can be achieved on medium length lines (3000 ft) and 6 Mbps can be achieved on long lines (4500 ft) [10]. Severe attenuation at high frequencies produces strong intersymbol interference in a twisted-pair channel. Discrete multitone (DMT) is the modulation technique considered in this paper.

In the twisted-pair environment, the primary noise sources are the crosstalk interference from the neighboring lines within the same binder that carry other VDSL transmissions or other data services such as ADSL or ISDN. The interference sources can be assumed Gaussian, but the crosstalk coupling transfer functions are frequency dependent. Each interference source produces the near-end crosstalk (NEXT) coupled into receivers located at the same side as the source and the far-end crosstalk (FEXT) coupled into receivers located on the opposite end of the twisted-pair. A typical transmission situation is illustrated in Fig. 6. NEXT interference is usually much larger than FEXT and the presence of NEXT in VDSL is often large enough to prevent VDSL transmission entirely. So, practical systems are designed so that all VDSL modems transmit in the same direction at the same frequency and time slot. This avoids NEXT entirely and leaves FEXT as the predominant noise source. The FEXT interference can be calculated by assuming that all interfering modems are transmitting at the worst-case power-spectral-density (PSD) limit.

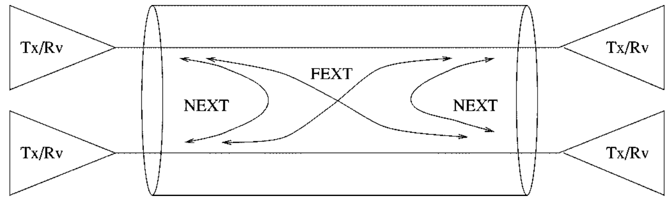

Fig. 6. Crosstalk in digital subscriber lines.

To allow bi-directional communication, a duplex scheme has to be used to coordinate the transmissions in the two directions. Time-division duplex (TDD) and frequency-division duplex (FDD) are typical duplex methods. In time-division duplex, the two modems transmit at alternating time slots and in frequency-division duplex, the two modems transmit at different frequency bands. The implementation of time-division duplex requires precise synchronization between the two modems. The latency in twisted-pairs could result a performance loss of up to 20%. For this reason, frequency-division duplex was chosen in the VDSL standard [11]. Frequency-division duplex usually requires guard bands between the upstream and the downstream bands. However, if a DMT system is used, it is possible to insert a cyclic suffix to allow the synchronization of the upstream and the downstream DMT-symbols. Such symbol-level synchronization allows the upstream and the downstream modems to use an arbitrary set of nonoverlapping DMT tones, thus making arbitrary frequency assignment possible with no guard bands in between [12]. The flexibility of the DMT system raises the following question. With frequency-division duplex, which set of tones should be assigned to the upstream transmission and which set of tones should be assigned to the downstream transmission? In fact, a range of services that require different upstream and downstream transmission rates are often desired in VDSL deployment, so it is desirable to find the exact tradeoff between the upstream capacity and the downstream capacity in a twisted-pair. The upstream and the downstream twisted-pair channels usually have the same frequency response. However, the two directions experience different crosstalk interference, so the effective channel-gain-to-noise ratios are different. Intuitively, each frequency bin should be assigned to the direction with less noise, but the exact frequency assignment also depends on the target rates. Previous attempts to solve this problem resorted to an exhaustive search [13], [14]. The entire

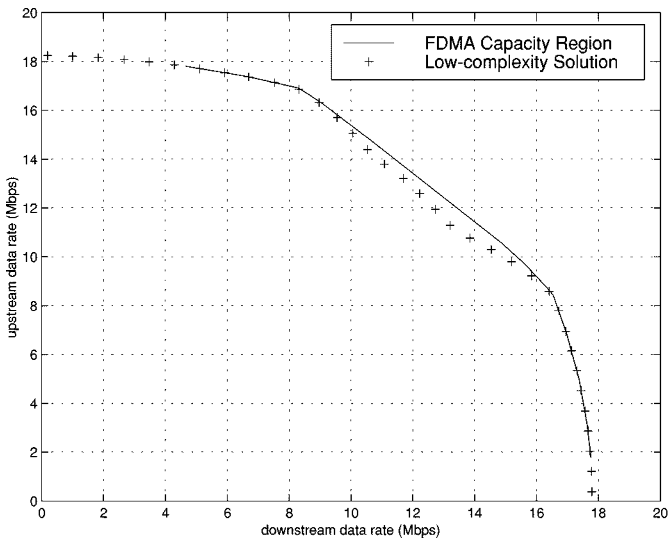

Fig. 7. The achievable rate region with frequency-division duplex in a 500-m VDSL line.

frequency band is divided into N bands and all 2^N possible upstream-downstream combinations are tried. Such an exhaustive search is exponentially complex and a granularity beyond $N = 32$ is not feasible within a reasonable computation time. However, because the upstream and the downstream transmissions do not interfere into each other in a FDD-DMT system, it is easy to observe that the optimal frequency duplex problem is just the FDMA-capacity problem for the mulitple-access channel. Thus, both the convex optimization approach and the low-complexity algorithm presented in previous sections can be used to solve this problem efficiently.

The capacity region using the frequency-division duplex scheme in a typical VDSL environment is presented in the following as an example. The simulation is performed on a 26-gauge 500 m copper twisted-pair with the standard ANSI noise B which includes ADSL, ISDN, HDSL and T1 crosstalk sources [15] and 20 VDSL far-end crosstalk sources computed using the standard FEXT coupling function where the coupling increases with frequency as $f^{3/2}$ [16]. A total power constraint of 11.5 dBm and a power-spectral-density constraint around -50 dBm/Hz are imposed on all modems [10]. The frequency range of 0 to 17.6 MHz is used, with the frequency spectrum divided into 256 bins. The target probability of error is 10^{-7}. An uncoded QAM transmission scheme at the probability of error 10^{-7} has an SNR gap of 9.8 dB from the Shannon capacity. The SNR gap is a concept that connects the information theoretical channel capacity with practical modulation and coding methods. A gap of 9.8 dB means that to achieve the channel capacity using uncoded QAM transmission, an extra 9.8 dB of power is needed. With error correcting codes, the gap is reduced by the coding gain. For the simulation purpose, a coding gain of 3.8 dB is assumed. In practice, to protect the system from nonstationary interference such as impulsive noise, an additional noise margin of 6 dB is often included. So, the effective gap assumed in this simulation is 9.8 dB $+$ 6 dB $-$ 3.8 dB $=$ 12 dB. This can be thought of as a 12 dB increase in the noise power-spectral-density. Fig. 7 shows the achievable upstream and downstream rate region for a 500 m VDSL line. The solid line represents the rate region obtained by maximizing $\alpha_1 R_{\rm up} + \alpha_2 R_{\rm down}$ for various values of (α_1, α_2) using the convex optimization approach. A nonlinear programming package MINOS [17] is used to obtain the optimal frequency partition. The crosses are obtained by searching through all two-band partitions as suggested by the low-complexity algorithm in Section III. In this example, the sub-optimal solution achieves at least 95% of the capacity in all cases. This numerical example is typical in the VDSL optimal frequency duplex problem.

In real systems, considerations other than data rate are also important. Practical systems often have VDSL lines that carry different upstream and downstream rates co-existing in the same binder. The optimal frequency partition obtained above is, however, different for different rate combinations in the rate region. This poses a problem because directly mixing them in the same binder would unduly create near-end crosstalk. In practice, therefore, it is necessary to find a universal partition that is not necessarily optimal for each individual line, but would represent a compromise among all service requirements [18]. The universal plan also has to be robust over all line configurations. Line impairments such as bridged-taps and radio interference have to be taken into account in designing an optimal frequency partition that works in all cases [19]. Also important is the practical requirement that the VDSL frequency plan is compatible with existing services such as ISDN and ADSL, so that VDSL does not emit unacceptable interference in the transmission bands of other services. All these considerations have to be taken into account in the design of a universal frequency plan. An acceptable band plan is often found by a combination of engineering intuition and exhaustive search. For this reason, the convex programming approach taken in this paper is most valuable not in providing a numerical solution to the optimal frequency partition problem for a specific situation, but in providing insights into the structure of the optimal solution for a class of situations. The numerical solution itself is most useful as a theoretical upper bound and a starting point for finding the right compromise among the practical considerations.

V. Conclusion

This paper proposes a numerical solution to the FDMA-capacity region problem for a Gaussian mulitple-access channel with intersymbol interference. The discrete frequency bin allocation problem is shown to have a convex programming relaxation, thus allowing the optimal frequency partition to be found with efficient numerical methods. A low-complexity bin-allocation method for the two-user system is also proposed. The algorithm explores the problem structure and the solution is near-optimal when the two channels are identical, or when the signal-to-noise ratio is high. The run-time complexity is $O(N \log N)$, where N is the total number of frequency bins. These numerical algorithms are then used to solve the optimal frequency duplex problem in VDSL. The duplex problem is posed as an FDMA-capacity problem for the mulitple-access channel and numerical examples are presented to illustrate the feasibility of the proposed solution in this context.

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Wei Yu received the B.A.Sc. degree in computer engineering and mathematics from the University of Waterloo in 1997 and M.S. in electrical engineering from Stanford University, Stanford, CA, in 1998. He is currently pursuing the Ph.D. degree at Stanford University.

His current research interests include communication theory, multiuser information theory, OFDM systems, and DSL systems.

John M. Cioffi (S'77–M'78–SM'90–F'96) received the B.S.E.E. degree in 1978 from the University of Illinois at Urbana-Champaign and the Ph.D. degree in electrical engineering from Stanford University, Stanford, CA, in 1984.

He was with Bell Laboratories from 1978 to 1984; IBM Research from 1984 to 1986. He has been with Stanford University from 1986 to the present as a Professor of electrical engineering. He founded Amati Com. Corporporation in 1991 (purchased by TI in 1997) and was officer/director from 1991 to 1997. He currently is on the boards or advisory boards of BigBand Networks, Coppercom, GoDigital, Ikanos, Ionospan, Ishoni, IteX, Marvell, Kestrel, Charter Ventures, and Portview Ventures and a member of the U.S. National Research Council's CSTB. His specific interests are in the area of high-performance digital transmission. He has published over 200 papers and holds over 40 patents, most of which are widely licensed, including basic patents on DMT, VDSL, and vectored transmission.

Dr. Cioffi has received various awards from the National Academy of Engineering 2001 and the IEEE Kobayashi Medal (2001), the IEEE Millennium Medal (2000), the IEE JJ Tomson Medal (2000), 1999 University of Illinois Outstanding Alumnus, 1991 IEEE COMMUNICATIONS MAGAZINE best paper, the 1995 ANSI T1 Outstanding Achievement Award, and the National Science Foundation Presidential Investigator (1987–1992).