Sum Capacity of a Gaussian Vector Broadcast Channel

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Abstract — **The sum capacity of a non-degraded Gaussian broadcast channel with multiple antennas at the transmitter and receivers is characterized under a certain condition. The sum capacity is achieved using a precoding technique for channels with transmitter side information. The optimal precoder has the structure of a decision-feedback equalizer.**

I. Introduction

Consider a discrete-time K-user broadcast channel:

$$\mathbf{y}_i = H_i \mathbf{x} + \mathbf{n}_i, \quad i = 1, \cdots K, \tag{1}$$

where $\mathbf{x}$ and $\mathbf{y}_i$'s are vector signals, H_i's are matrix channels, and $\mathbf{n}_i$'s are Gaussian vector noises. Independent information is to be sent to each user. This paper aims to characterize the sum capacity of this non-degraded broadcast channel.

The Gaussian vector broadcast channel is related to the following channel with transmitter side information: $\mathbf{y} = \mathbf{x}+\mathbf{s}+\mathbf{n}$, where $\mathbf{n}$ is the Gaussian noise, and $\mathbf{s}$ is the Gaussian interference which is known to the transmitter but not known to the receiver. In a surprising result, Costa [1] showed that the capacity of this channel is the same as if $\mathbf{s}$ does not exist. Thus, interference known at the transmitter can be effectively "pre-subtracted". This result, when generalized to vector channels [2], gives the following precoding strategy for the broadcast channel. Set $\mathbf{x} = \mathbf{x}_1 + \cdots + \mathbf{x}_K$. Let $\mathbf{x}_1$ transmit to $\mathbf{y}_1$ regarding $\mathbf{x}_2 \cdots \mathbf{x}_K$ as noise. Let $\mathbf{x}_2$ transmit to $\mathbf{y}_2$ regarding $\mathbf{x}_3 \cdots \mathbf{x}_K$ as noise, but regarding $\mathbf{x}_1$ as known interference which can be pre-subtracted. Likewise, $\mathbf{x}_3$ may transmit to $\mathbf{y}_3$ while having $\mathbf{x}_1$ and $\mathbf{x}_2$ pre-subtracted, etc.

This precoding technique generalizes Caire and Shamai's work [3] on the sum capacity of two-user two-antenna broadcast channels. This paper shows that precoding achieves the sum capacity of a broadcast channel with an arbitrary number of users and an arbitrary number of antennas under a certain condition.

II. Sum Capacity

The sum capacity of a broadcast channel is clearly bounded by the capacity of the channel where receivers cooperate: $\max_{S_x} I(\mathbf{X};\mathbf{Y}_1,\cdots,\mathbf{Y}_K)$, where S_x denotes the transmit covariance matrix. Further, as observed in [4], the broadcast channel capacity does not depend on the correlation among $\mathbf{n}_i$. So, $C_{sum} \le \min_{S_n} \max_{S_x} I(\mathbf{X};\mathbf{Y}_1,\cdots,\mathbf{Y}_K)$, where S_n is the covariance matrix of the noise vector $\mathbf{n}^T = [\mathbf{n}_1^T \cdots \mathbf{n}_K^T]$, and the minimization is over all possible noise correlations.

This outer bound turns out to be tight. To show this, fix S_x, and find the S_n that minimizes $\frac{1}{2}\log|HS_xH^T+S_n|/|S_n|$, over all possible noise correlations. ($H^T = [H_1^T, \cdots H_K^T]$.) The mutual information is convex in S_n. The Karush-Kuhn-Tucker (KKT) condition for the minimization problem has a diagonal form, if the minimizing S_n is non-singular:

$$S_n^{-1} - (HS_xH^T + S_n)^{-1} = \mathrm{diag}\{\Psi_1, \cdots, \Psi_K\}. \tag{2}$$

Next, assume the worst-case noise correlation, and design an optimal decision-feedback receiver [6] for the vector channel as if $\mathbf{y}_1, \cdots \mathbf{y}_K$ cooperate. The decision-feedback receiver consists of a feedforward matrix and a feedback matrix. Interestingly, when the noise covariance has the worst-case correlation, it is possible to find an appropriate linear pre-processing matrix for $\mathbf{x}_i$'s so that the feedforward matrix becomes diagonal. This is a consequence of the diagonalization condition (2), and it implies that only individual processing of $\mathbf{y}_i$ is required at the feedforward stage. Further, the decision-feedback operation, which successively subtracts interference at the receiver, can be implemented at the transmitter using interference pre-subtraction. Thus, with the worst-case noise correlation, an optimal precoder can be designed so that it requires no cooperation whatsoever among $\mathbf{y}_1, \cdots \mathbf{y}_K$, and achieves the same sum rate as a decision-feedback receiver. Since the decision-feedback receiver achieves $I(\mathbf{X};\mathbf{Y}_1,\cdots,\mathbf{Y}_K)$, this implies that $\min_{S_n} I(\mathbf{X};\mathbf{Y}_1,\cdots,\mathbf{Y}_K)$ is achievable for the broadcast channel. Further, one may maximize over all S_x, so $C_{sum} \ge \max_{S_x} \min_{S_n} I(\mathbf{X};\mathbf{Y}_1,\cdots,\mathbf{Y}_K)$.

$I(\mathbf{X};\mathbf{Y}_1,\cdots,\mathbf{Y}_K)$ is concave in S_x, and convex in S_n. The constraints are convex. So, minimax equals maxmin [5], and the converse equals the achievability. In fact, the broadcast channel sum capacity can be interpreted as the Nash equilibrium of a game, where a signal player chooses a transmit covariance matrix to maximize the mutual information, and "nature" chooses a noise correlation to minimize the mutual information. When receivers do not cooperate, "nature" conspires to give effectively a worst possible noise.

Theorem 1 *The sum capacity of a Gaussian vector broadcast channel is a saddle-point of* $\frac{1}{2}\log|HS_xH^T+S_n|/|S_n|$, *if* S_n *at the saddle-point is non-singular. The saddle-point is computed with the following constraints:* $\mathrm{trace}(S_x) \le P$ *and the ith block diagonal entry of* S_n *is equal to the covariance matrix of* $\mathbf{n}_i$.

References

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