Competitive Equilibrium in the Gaussian Interference Channel

Wei Yu[1] and John M. Cioffi
Electrical Engineering Department
Stanford University, Stanford, CA 94305, U.S.A.
e-mail: weiyu@stanford.edu

Abstract — **A game theory formulation for the Gaussian interference channel is given under the assumption that no interference subtraction is performed. The existence, uniqueness and stability of a pure strategy Nash equilibrium is established under a mild-interference condition,**

I. Introduction

In a Gaussian interference channel, two independent sender-receiver pairs attempt to communicate in the presence of interference from each other. The capacity region for the interference channel is still an open problem. The largest rate region presently known is achieved with superposition coding and interference subtraction [1], but its optimality is not yet known. However, the traditional view of the interference channel allows the two senders, while remaining independent, to be cooperative in their respective coding strategies. If such cooperation cannot be assumed, the interference channel becomes a non-cooperative game. This paper studies the interference channel from this game theory perspective. We focus on Gaussian interference channels with memory, but make the simplifying assumption that no interference subtraction is performed regardless of interference strength. We ask the following questions: if each sender's sole objective is to maximize its *own* data rate, can an equilibrium be achieved in a competitive environment? If so, is such an equilibrium unique?

II. Competitive Equilibrium

The two senders in a Gaussian interference channel,

$$\mathbf{y}_1 = \mathbf{x}_1 + A_2\mathbf{x}_2 + \mathbf{n}_1 \tag{1}$$

$$\mathbf{y}_2 = \mathbf{x}_2 + \mathcal{A}_1\mathbf{x}_1 + \mathbf{n}_2, \tag{2}$$

are considered as two players in a game. The channel transfer functions are normalized to unity. The square magnitude of the interference transfer functions $\mathcal{A}_1$ and $\mathcal{A}_2$ are denoted as $\alpha_1(f)$ and $\alpha_2(f)$. Let $N_1(f)$ and $N_2(f)$ denote noise power spectrum densities. The structure of the game, i.e., the interference coupling functions and noise power, are common knowledge to both players. A strategy[2] for each player is its transmit power spectrum, $P_1(f)$ and $P_2(f)$, subject to the power constraints $\int_0^F P_1(f)df \le \mathbf{P}_1$, and $\int_0^F P_2(f)df \le \mathbf{P}_2$, respectively. The payoffs are data rates :

$$R_1 = \int_0^F \log\left(1 + \frac{P_1(f)}{N_1(f) + \alpha_2(f)P_2(f)}\right) df, \tag{3}$$

$$R_2 = \int_0^F \log\left(1 + \frac{P_2(f)}{N_2(f) + \alpha_1(f)P_1(f)}\right) df, \tag{4}$$

where bandwidth up to F is used. The game is not zero-sum. We are interested in characterizing its pure strategy Nash equilibrium.

A Nash equilibrium is a strategy profile in which each player's strategy is an optimal response to the other player's strategy [2]. For the interference channel, the optimal response for a player is the waterfilling of its power with respect to the combined noise and interference. If the power distributions are such that waterfilling is achieved simultaneously for both players, a Nash equilibrium is reached. At a Nash equilibrium, neither player has an incentive to move away from its present power distribution.

Theorem: Suppose that $\sup_f \alpha_1(f) \cdot \sup_f \alpha_2(f) < 1$, then a pure strategy Nash equilibrium in the Gaussian interference game exists, is unique, and is stable.

Proof: The first idea is to recognize that if $\alpha_1(f)\cdot\alpha_2(f) < 1$ $\forall f$, there is a Nash equilibrium corresponding to every water level (K_1, K_2). For fixed (K_1, K_2), the Nash equilibrium $(P_1(f), P_2(f))$ is found by solving the waterfilling condition:

$$P_1(f) + \alpha_2(f)P_2(f) + N_1(f) = K_1 \tag{5}$$

$$P_2(f) + \alpha_1(f)P_1(f) + N_2(f) = K_2, \tag{6}$$

unless $\alpha_1(f) > \frac{K_2 - N_2(f)}{K_1 - N_1(f)}$, in which case $P_1(f) = \max\{0, (K_1 - N_1(f))\}$ and $P_2(f) = 0$, or $\alpha_2(f) > \frac{K_1 - N_1(f)}{K_2 - N_2(f)}$, in which case $P_2(f) = \max\{0, (K_2 - N_2(f))\}$ and $P_1(f) = 0$.

Next, we establish that for a given power constraint $(\mathbf{P}_1, \mathbf{P}_2)$, there exists (K_1, K_2) whose Nash equilibrium has exactly this power. For each (K_1, K_2), denote the power level at the corresponding Nash equilibrium as $(\mathbf{P}_{K_1}, \mathbf{P}_{K_2})$. Observe that when $\alpha_1(f)\cdot\alpha_2(f) < 1$, if $K_1 < K_1'$ and $K_2 = K_2'$, then $\mathbf{P}_{K_1} \le \mathbf{P}_{K_1'}$ and $\mathbf{P}_{K_2} \ge \mathbf{P}_{K_2'}$. Now, start with $K_1 = K_2 = 0$. Increase K_1 until $\mathbf{P}_{K_1} = \mathbf{P}_1$, then increase K_2 until $\mathbf{P}_{K_2} = \mathbf{P}_2$. But then, we have $\mathbf{P}_{K_1} \le \mathbf{P}_1$ by observation. So, we can increase K_1 again, until $\mathbf{P}_{K_1} = \mathbf{P}_1$, then increase K_2, etc. The increasing sequences of K_1's and K_2's converge because they cannot go to infinity with finite power constraints. The limit point is a Nash equilibrium corresponding to $(\mathbf{P}_1, \mathbf{P}_2)$.

To prove uniqueness, let $(P_1^N(f), P_2^N(f))$ be the power distribution at a Nash equilibrium. Start with any power distribution $P_1^{(0)}(f)$ that satisfies the power constraint. Waterfill for $P_2^{(0)}(f)$, assuming $P_1^{(0)}(f)$ as interference. Then, waterfill for $P_1^{(1)}(f)$, assuming $P_2^{(0)}(f)$ as interference, etc. This iterative waterfiling process coverges in $\mathbf{L}_1$-norm $\int_0^F |P_1^{(k)} - P_1^N| df$ because $\max(||(P_1^{(k+1)} - P_1^N)_+||_1, ||(P_1^{(k+1)} - P_1^N)_-||_1) \le \sup\alpha_2(f) \cdot \max(||(P_2^{(k)} - P_2^N)_+||_1, ||(P_2^{(k)} - P_2^N)_-||_1) \le \sup\alpha_2(f)\sup\alpha_1(f)\cdot\max(||(P_1^{(k)} - P_1^N)_+||_1, ||(P_1^{(k)} - P_1^N)_-||_1)$, which is a contraction by the assumption that $\sup\alpha_1(f)\cdot\sup\alpha_2(f) < 1$. So, $P_1^{(k)} \to P_1^N$ in $\mathbf{L}_1$-norm as $k \to \infty$. □

References

[1] T.S. Han, K. Kobayashi, "A New Achievable Rate Region for the Interference Channel," *IEEE Trans. Info. Theory*, vol. 27, pp. 49–60, January 1981.

[2] D. Fudenberg and J. Tirole, *Game Theory*, MIT Press, 1991.

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[2] Only pure (or deterministic) strategy is considered here.