FDMA Capacity of the Gaussian Multiple Access Channel with ISI

Wei Yu and John M. Cioffi
Electrical Engineering Department
Stanford University, Stanford, CA 94305, U.S.A.
email: weiyu@stanford.edu

Abstract— **This paper proposes a numerical method for characterizing the achievable rate region for a Gaussian multiple access channel with ISI under the frequency division multiple access restriction. The frequency spectrum is divided into discrete frequency bins and the discrete bin assignment problem is shown to have a convex programming relaxation, making it tractable to numerical algorithms. The run-time complexity may be further reduced in the two-user case if the two channels are identical, or if the signal-to-noise ratio is high.**

I. Introduction

In a Gaussian multiple access channel, M independent senders attempt to communicate with a single receiver in the presence of additive Gaussian noise. The capacity region of the Gaussian multiple access channel is the set of rates $(R_1, R_2, ..., R_M)$ at which the receiver may decode information from each sender without error. In general, each user has a different channel, and each channel has intersymbol interference (ISI). The capacity region for such multiple access channel with ISI was solved by Cheng and Verdu in [1], where they used a multiuser waterfilling scheme to allocate each user's power over frequency. They showed that the optimum multiuser waterfilling spectrum involves superposition in frequency, so frequency division multiple access (FDMA) alone is not optimal except in special cases. However, FDMA may be desirable in practical implementations, especially in orthogonal frequency division multiplex (OFDM) systems. This paper addresses the problem of finding the achievable rate region and finding the optimal frequency partition for multiple access channels with ISI if the frequency division multiple access restriction is placed.

The FDMA-capacity problem is motivated by the search for the optimum frequency duplex scheme for VDSL (Very-high-speed Digital Subscriber Lines). In VDSL, frequency division duplex is used to separate upstream and downstream transmission. The optimal frequency assignment between upstream and downstream is an optimal FDMA-capacity problem for a multiple access channel with two users. Previously, [2] studied a similar joint signaling problem in the crosstalk environment under the assumption that two channels are the same. In the present optimal frequency duplex problem, however, upstream and downstream often have different noise disturbances, so a multiple access channel model with a different channel for each user is more appropriate. The FDMA-capacity problem also appears in the wireless OFDM context. For example, [3] studied a power-minimization problem with rate constraints, which is a dual of the present capacity problem. Finally, of practical interests are sub-optimal low complexity algorithms. In this direction, [4] and [5] showed that in the setting of providing different quality of service (QoS) for different services in a single subscriber line, a simple search algorithm to find the optimal partition of OFDM tones is asymptotically optimal. This motivated our search for low complexity alternatives in the current formulation.

The rest of the paper is organized as follows. In section II, the FDMA-capacity problem for the multiple access channel is posed as a convex programming problem. Section III shows that under two special cases the optimal two-user FDMA partition is a two-band partition. Section IV describes the VDSL duplex problem as a practical example. Conclusions are drawn in section V.

II. Optimal Frequency Partition

A Gaussian multiple access channel with M users is shown in Fig. 1. The goal is to solve for the rate region achievable with FDMA. The rate region characterizes the trade-off among data rates for different users. The point on the boundary of the rate achievable region where the normal to the tangent hyperplane is in the direction of the vector $(\alpha_1, \alpha_2, ..., \alpha_M)$, where $\alpha_i \geq 0$, can be found by maximizing $\sum_{i=1}^{M} \alpha_i R_i$. Maximizing this aggregate data rate for various α_i's characterizes the entire rate region. The variables $\{\alpha_i\}$ can also be interpreted as the relative priorities given to each user.

In the following, we concentrate on the multiple access channel with two users. To cast the problem of maximizing the aggregate data rate with FDMA constraint into a finite dimensional problem, we divide the frequency spectrum into a large number of rectangular frequency bins, and assume that the channel response is flat within each bin. As the number of bins increases to infinity, the piecewise constant channel model approaches the actual channel. In this discretized version, each frequency bin may be assigned to one of the two users, so the optimal frequency division problem becomes a set assignment problem. However, the set assignment problem is an integer programming problem, which is computationally dif-

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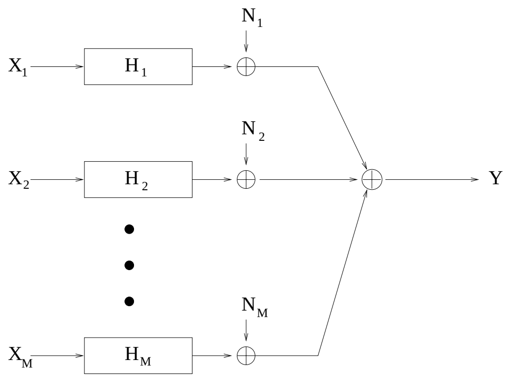

Fig. 1. A multiple access channel

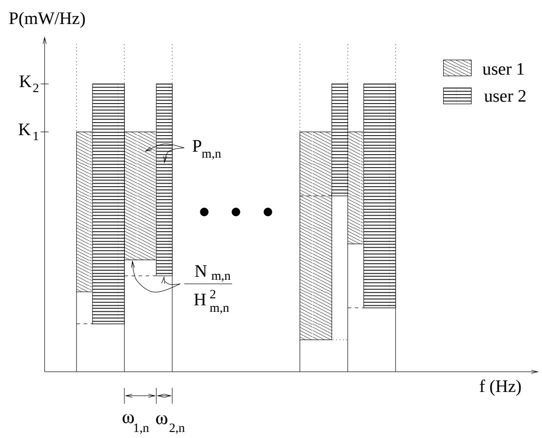

Fig. 2. Optimization with frequency division multiple access

ficult. So, instead of restricting the boundaries of frequency partition between the two users to align with the bin boundaries as the integer programming does, we allow the boundary to be anywhere in the bin, hence relaxing the integer programming problem into a continuous-variable optimization problem. In effect, each frequency bin may be sub-partitioned arbitrarily between the two users as illustrated in Fig. 2. As the bin width becomes small, at the optimum, almost every bin except a few will be exclusively assigned to one user. This is because the optimal allocation usually involves only a few frequency bands. When the width of the frequency bin is less than the width of the narrowest frequency band in the optimal allocation, frequency bins in the middle of the bands are fully allocated to one user only, and those on the boundary of the frequency bands are the only ones shared. In practice where an integer solution is desired, the boundary bins can be assigned to either user arbitrarily. As long as there are only small number of such shared bins, their assignment does not affect the total rate appreciably.

The discrete partition of the frequency spectrum into flat subchannels is similar to the OFDM setting where an IFFT/FFT pair together with cyclic prefix are used to perform modulation and de-modulation. Each OFDM tone corresponds to a frequency bin. However, frequency partition by FFT results in significant adjacent band energy, so there is no easy way to divide each tone further in frequency. Such sub-division is crucial to form a relaxation to the original integer programming problem.

Mathematically, the optimization problem can be posed as follows. Let $\omega_{m,n}$ and $P_{m,n}$ be the amount of bandwidth and power assigned to user m in the nth frequency bin respectively. The objective is to choose $\omega_{m,n}$ and $P_{m,n}$ to maximize the aggregate data rate:

$$\begin{aligned} \max \quad & \sum_{m=1}^{M} \alpha_m \cdot \sum_{n=1}^{N} \omega_{m,n} \log\left(1 + \frac{P_{m,n} \cdot H_{m,n}^2}{\omega_{m,n} \cdot N_{m,n}}\right) \\ \text{s.t.} \quad & \sum_{m=1}^{M} \omega_{m,n} \leq \omega_n, \forall n, \\ & \sum_{n=1}^{N} P_{m,n} \leq P_m, \forall m, \\ & P_{m,n}/\omega_{m,n} \leq S_{m,n}, \\ & P_{m,n} \geq 0, \\ & \omega_{m,n} \geq 0, \end{aligned} \tag{1}$$

where N is the total number of frequency bins, M is the total number of users, α_m are the relative weights given to each user, $H_{m,n}$ and $N_{m,n}$ are the channel response and noise power spectral density respectively for user m in frequency bin n, and ω_n is the width of the nth frequency bin. A total power constraint P_m (in dBm), and a power-spectral-density constraint $S_{m,n}$ (in dBm/Hz) are imposed on each user. Solving the optimization problem for various α_m's traces the boundary of the achievable rate region.

In general, constrained nonlinear optimization problems are not easy to solve because many local minima potentially exist. However, in this case, two features make the problem tractable. First, as a function of $(P_{m,n}, \omega_{m,n})$, the objective function is concave. A proof of this fact is presented in the appendix. Secondly, the constraints are linear which means the constraint set is convex. Since the optimization problem has the form of a concave function constrained to a convex set, it is a convex programming problem. Thus, a local maximum is also a global maximum, and numerical convex programming algorithms such as the interior-point method are well-suited to solve this problem efficiently [6]. The complexity of numerical convex programming methods increases as a polynomial function of the number of variables.

III. Low Complexity Solutions

In convex programming, it is usually possible to explore the specific problem structure to reduce dimensionality and run-time complexity. Previously, [4] and [5] proposed a searching method to solve the optimal allocation problem to guarantee QoS for multiple services in a single subscriber line. Although the problem setting in the current case is different, the same search method applies under two special circumstances.

To gain some intuition, consider a two-user case where the channel and the power constraints are the same for the two users. No power-spectral-density limit is imposed and the objective is to maximize $R_1 + R_2$. In this special case, [1] showed that an optimal solution is to waterfill with combined power, so that the water levels for the two users are equal. There are many methods to achieve the maximum rate sum. For example, we can divide every frequency bin equally into two halves and assign one half to each user. Or, we can order the sub-channels according to channel-gain-to-noise ratio, and divide the sorted channels into two bands. As long as the frequency band boundary is chosen so that the two users have the same water level, maximum combined data rate is achieved, regardless of which user is assigned the better sub-channels.

When the priorities of the two users are different, the two users no longer have the same water level at the optimum. Nevertheless, the two-band partition remains the optimal one when the two channels are identical.

Theorem 1: Consider a Gaussian multiple access channel with two users, each with a total power constraint, but no power-spectral-density constraint. Define the channel-gain-to-noise ratio as $g_{m,n} = H_{m,n}^2/N_{m,n}$, where $m = 1, 2$, and $n = 0 \cdots N$. Suppose that the channel and noise characteristics for the two users are identical, i.e. $g_{1,n} = g_{2,n} = g_n, \forall n$. Suppose further that g_n is monotonically decreasing in n, i.e., $g_i \geq g_j$ for $i < j$. Then, the optimal frequency partition maximizing $R_1 + \alpha R_2$, where $\alpha > 1$, consists of two frequency bands only. More precisely, at the optimum, there exist L_1 and L_2, $1 \leq L_1 \leq L_2 \leq N$, such that $\omega_{1,l} = 0$ and $\omega_{2,l} = 1$ for all $l < L_1$, and $\omega_{1,l} = 1$ and $\omega_{2,l} = 0$ for all $L_1 < l < L_2$. Frequency bins beyond L_2 are not used by either user. Only the variables ω_{1,L_1}, ω_{2,L_1} and ω_{1,L_2}, ω_{2,L_2} may take fractional values between 0 and 1.

The proof is presented in the appendix. The proof depends on the Karush-Kuhn-Tucker (KKT) condition for the optimization problem (1). In (1), there is a potential singularity when $\omega_{n,m} = 0$, so instead of using the constraint $\omega_{n,m} \geq 0$, we use the constraint $\omega_{n,m} \geq \epsilon$, where ϵ is positive but much less than ω_n. Then, the KKT condition asserts the following. There exist positive constants K_m, such that $\forall n = 1 \cdots N, \forall m = 1, 2$,

$$\frac{P_{m,n}}{\omega_{m,n}} + \frac{1}{g_{m,n}} = K_m, \text{ when } P_{m,n} > 0, \tag{2}$$

and, $1/g_{m,n} \geq K_m$, when $P_{m,n} = 0$. Also, $\forall n$,

$$\log\left(1 + \frac{P_{1,n} \cdot g_{1,n}}{\omega_{1,n}}\right) - \frac{P_{1,n} \cdot g_{1,n}/\omega_{1,n}}{1 + P_{1,n} \cdot g_{1,n}/\omega_{1,n}} =$$
$$\alpha \log\left(1 + \frac{P_{2,n} \cdot g_{2,n}}{\omega_{2,n}}\right) - \alpha\frac{P_{2,n} \cdot g_{2,n}/\omega_{2,n}}{1 + P_{2,n} \cdot g_{2,n}/\omega_{2,n}}, \tag{3}$$

when $\omega_{1,n} > \epsilon$ and $\omega_{2,n} > \epsilon$. When $\omega_{1,n} = \epsilon$, the left-hand side is strictly less than the right-hand side. When $\omega_{2,n} = \epsilon$, the inequality is reversed. Because the problem is concave, (2) and (3), together with the total power constraints $\sum_{n=1}^{N} P_{m,n} \leq P_m$, total bandwidth constraints $\sum_{m=1}^{M} \omega_{m,n} \leq \omega_n$, and the positivity constraints on $P_{m,n}$ and $\omega_{n,m}$ are the necessary and sufficient optimality conditions.

Equation (2) is the classical waterfilling equation. Within each user, power is distributed according to the waterfilling algorithm among the used sub-channels. Equation (3) decides how much bandwidth is given to each user. (2) and (3) are nonlinear, and there is no analytic solution in general. However, when the conditions of Theorem 1 are satisfied, all frequency bins except one are assigned completely to either user 1 or user 2 as two bands. Only the single boundary bin is shared. When the number of frequency bins is large, the data rate contributed by one frequency bin is small. The boundary bin may be arbitrarily assigned to user 1 or user 2, and the discrete assignment is very close to the true optimum. This suggests the following algorithm to find a near-optimal discrete partition.

Sort all the g_n's from the largest to smallest. For each $n = 0 \cdots N$, waterfill for user 2 using frequency bins 1 to n, waterfill for user 1 using frequency bins $n + 1$ to N. Compute the aggregate data rate. The optimal partition boundary is the n which gives the maximum aggregate data rate. Frequency bins from 1 to n are exclusively used by user 2, and frequency bins from $n + 1$ to N are exclusively used to user 1. The run-time complexity of the algorithm is $O(N \log N)$, because sorting takes $O(N \log N)$ operations, each waterfill takes $O(N)$ operations on sorted channels, and there are at most $O(\log N)$ waterfilling to do with a binary search on the boundary bin. Binary search is feasible because for each bin n, equation (3) can be used to decide whether the optimal boundary is higher or lower than n.

There is another special case where a two-band partition is optimal. This happens when the signal-to-noise ratio is high.

Theorem 2: Consider a Gaussian multiple access channel with two users, each with a total power constraint, but no power-spectral-density constraint. Define the signal-to-noise ratio, $\text{SNR}_{m,n} = P_{m,n} g_{m,n}/\omega_{m,n}$. Assume that at the optimum, SNR is much larger than 1 (0dB) in all frequency bins. Further assume that the normalized difference of the two users' channel-gain-to-noise ratios (in dB) is a monotonic function of the frequency. More precisely, assume $g_{1,n}/g_{2,n}^{\alpha}$ is monotonic in n. Then, the optimal frequency partition maximizing $R_1 + \alpha R_2$ consists of two frequency bands only.

The proof of Theorem 2 is also presented in the appendix. Since the solution is two-band, the previous low complexity algorithm applies with only a slight modification. The complexity is also $O(N \log N)$. The same algorithm was

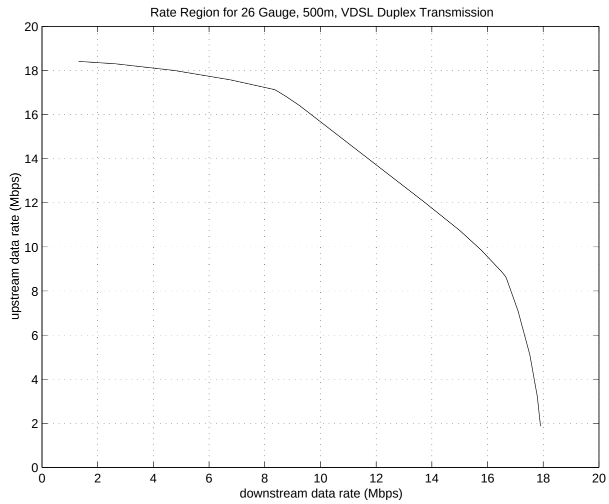

Fig. 3. Achievable rate region with frequency division duplex in a VDSL line

noted in [7] for the special case where $\alpha = 1$.

IV. Numerical Example

The optimal duplex problem in VDSL is used as a numerical example. In a frequency division duplex system, upstream and downstream do not interfere into each other, so the optimal duplex problem is the FDMA-capacity problem for a multiple access channel with two users. Arbitrary frequency division of upstream and downstream can be implemented in practice with Discrete Multitone (DMT) modulation when cyclic suffix and cyclic prefix are added [8].

Simulation is performed on a 26-gauge 500m cable with standard NEXT and FEXT noise models. The total power and power-spectral-density constraints are as defined in [9]. 6dB margin and 3.8dB coding gain are assumed. The target probability of error is 10^{-7}. Fig. 3 shows the achievable upstream and downstream data rates. The rate region is obtained by maximizing $R_{up} + \alpha R_{down}$ for various values of α using the nonlinear programming package MINOS [10]. The trade-off between upstream and downstream data rates is clearly illustrated.

V. Conclusion

We numerically solved for the FDMA capacity region for a Gaussian multiple access channel with ISI. The discrete frequency-bin allocation problem was shown to have a convex programming relaxation, allowing optimal frequency partition to be found with efficient numerical methods. The run-time complexity may be further reduced to $O(N \log N)$ in the two-user case when the two channels are identical, or when the signal-to-noise ratio is high. Numerical examples are given for the optimal frequency division duplex problem in VDSL.

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Appendix

Proof that (1) is concave: To see that the objective function (1) is concave, observe that the objective is a positive linear combination of functions of the type $f(x,y) = x \log(1 + y/x)$, where $x \geq 0$ and $y \geq 0$. Since positive linear combination of concave functions is concave, to prove the concavity of the objective, we only need to show that $f(x,y)$ is concave in (x,y) in the first quadrant.

A two-dimensional function is concave if and only if its restriction to a line is concave [6]. Let $y = ax + b$, then

$$f(x,y) = x \log\left(1 + \frac{y}{x}\right) = x \log\left(1 + a + \frac{b}{x}\right).$$

Call the above function $g(x)$. The concavity of $g(x)$ is verified by taking its second derivative:

$$g''(x) = \frac{b/(1+a)}{x + b/(1+a)} \left(\frac{1}{x + b/(1+a)} - \frac{1}{x}\right).$$

Consider three cases:

- $b/(1+a) \geq 0$: Since x is non-negative, the first term in $g''(x)$ is non-negative. The second term is negative. So, the product is negative.
- $-x < b/(1+a) < 0$: The numerator of the first term is negative, but the denominator is positive, so the first term in $g''(x)$ is negative. The second term is positive because x is positive, and $b/(1+a)$ is negative. So, the product is negative.
- $b/(1+a) \leq -x$: In this case, both the numerator and the denominator of the first term is negative, so the first term is positive. The second term is negative since both fractions are negative. Again, the product is negative.

The second derivative is always negative. So, $g(x)$ is concave when $x \geq 0$, and $f(x,y)$ is concave in the first quadrant, and (1) is concave in $(P_{m,n}, \omega_{m,n})$. $\square$

Proof of Theorem 1: We start with the KKT condition, which is necessary and sufficient for optimality. To avoid singularity at zero, we constrain $\omega_{m,n} \geq \epsilon$. We need to prove that at the optimal power and bandwidth allocation, with K_1, K_2 as each user's respective optimal water levels, there exist L_1, L_2, such that the left-hand side of (3) is strictly less than the right-hand side for $1 \leq n < L_1$, and the left-hand side of (3) is strictly greater than the right-hand side for $L_1 < n < L_2$, and beyond L_2, $1/g_n \geq K_1$ and $1/g_n \geq K_2$.

If a frequency bin is shared between the two users, (2) and (3) need to be satisfied with equality. Substitute (2) into (3), we have:

$$\log(K_1 \cdot g_n) + \frac{1}{K_1 \cdot g_n} - 1 = \alpha \log(K_2 \cdot g_n) + \alpha \frac{1}{K_2 \cdot g_n} - \alpha. \quad (4)$$

This equality is satisfied in each shared bin. If a bin is to be assigned to user 2, the left-hand side of (4) is strictly less than the right-hand side. If a bin is to be assigned to user 1, the left-hand side of (4) is strictly greater. Consider the difference between the left-hand side and right-hand side as a function of $1/g_n$, call it $f(\cdot)$,

$$\begin{aligned} f(1/g_n) &= (\alpha-1)\log(1/g_n) + \left(\frac{1}{K_1} - \frac{\alpha}{K_2}\right)(1/g_n) \\ &\quad + \log K_1 - \alpha \log K_2 - 1 + \alpha. \end{aligned} \quad (5)$$

When $f(1/g_n) < 0$, the nth frequency bin is assigned to user 2. When $f(1/g_n) > 0$, the nth frequency bin is assigned to user 1. Therefore, when g_n's are sorted, every time $f(x)$ crosses zero, the frequency bin assignment switches from one user to the other.

Consider user 2 first. We will show that the frequency assignment for user 2 is a single band. To prove this, we only need to show that $f(x)$ has at most one root in the range $0 < x < K_2$. This is because user 2 only uses frequency bins where $1/g_n$ is less than its waterfill level K_2.

Without loss of generality, assume $\alpha > 1$. Consider three separate cases:

- $K_2/K_1 \geq \alpha$. In this case, $1/K_1 - \alpha/K_2 \geq 0$, and $\alpha - 1 > 0$ by assumption. Both $\log(x)$ and x are increasing function of x, hence $f(x)$ can only have one root.
- $1 < K_2/K_1 < \alpha$. In this case, $1/K_1 - \alpha/K_2 \leq 0$, By taking derivative of $f(x)$, it is easy to see that $f(x)$ is increasing until it reaches a maximum at $x_{max} = K_2 \cdot K_1(\alpha-1)/(\alpha K_1 - K_2)$, after which point, it is decreasing. But, $x_{max} = K_2 \frac{\alpha-1}{\alpha - K_2/K_1} \geq K_2 \geq K_1$. So, in the range where $x < K_2$, $f(x)$ is increasing, hence it can only have one root.
- $K_2/K_1 \leq 1$. Again, $1/K_1 - \alpha/K_2 \leq 0$, so $f(x)$ is increasing until x_{max}, then decreasing. In this case, $x_{max} < K_2$, so $f(x)$ can potentially decrease to zero after it reaches the maximum. But, the range of our interest is $x < K_2$, and it is easy to check that $f(K_2) = K_2/K_1 - \log(K_2/K_1) - 1 \geq 0$. So $f(x)$ can cross zero only once in the range $x < K_2$.

In all cases, $f(x)$ crosses zero-axis only once. Since g_n is sorted from the largest to the smallest, we can find L_1 such that $f(1/g_n) < 0$ for $n < L_1$, and $f(1/g_n) > 0$ for $n > L_1$. (If $f(1/g_0) > 0$, set $L_1 = 1$; if $f(1/g_N) < 0$, set $L_1 = N$.) So, frequency bins $1 \leq n < L_1$ are used exclusively by user 2. We can perform waterfilling on the rest of the frequency bins to determine the set of frequency bins to be used by user 1. Since only frequency bins beyond L_1 are available, and since g_n are sorted, user 1 will use bins from L_1 to some $L_2 \leq N$. Bins beyond L_2 are used by neither user. □

Proof of Theorem 2: Again, for the nth frequency bin be shared between the two users, (3) needs to be satisfied with equality, i.e.,

$$\begin{aligned} &\log(1+\mathrm{SNR_{1,n}}) - \frac{\mathrm{SNR_{1,n}}}{1+\mathrm{SNR_{1,n}}} = \\ &\alpha\log(1+\mathrm{SNR_{2,n}}) - \alpha\frac{\mathrm{SNR_{2,n}}}{1+\mathrm{SNR_{2,n}}}, \end{aligned} \quad (6)$$

where the signal-to-noise ratio $\mathrm{SNR}_{m,n}$ is defined to be $P_{m,n}H_{m,n}^2/\omega_{m,n}N_{m,n}$. The direction of the inequality determines whether the frequency bin is exclusively used by user 1 or user 2. Since SNR is assumed to be high, the second term on both sides of (6) can be approximated by 1. Let K_1 and K_2 be fixed to their respective values at the optimum. Substitute (2) into (6), take the difference between the left-hand side and the right-hand side of (6), call the function $f()$, as a function of $g_{1,n}$ and $g_{2,n}$,

$$f(g_{1,n}, g_{2,n}) \approx \log(K_1 \cdot g_{1,n}) - \alpha \log(K_2 \cdot g_{2,n}) + \alpha - 1. \quad (7)$$

We need to decide whether the above is greater than zero or less than zero for each n. This function can be rewritten as $f(g_{1,n}, g_{2,n}) = \log(g_{1,n}/g_{2,n}^{\alpha}) - \log(K_2^{\alpha}/K_1) + \alpha - 1$. Since log() is a monotonic function, and since $g_{1,n}/g_{2,n}^{\alpha}$ is sorted, $f(g_{1,n}, g_{2,n})$ is a monotonic function of n. Therefore, for some L_1, $f(\cdot)$ is less than zero for $n < L_1$, greater than zero for $n > L_1$, implying that the optimum partition is a two-band solution following the same argument as in the proof of Theorem 1. □

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