Degrees of Freedom Achieved Using Subspace Alignment Chains for Three-Cell Networks

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***Abstract*—In this paper we extend the notion of subspace alignment chains (SACs) for the three-user multiple-input multiple-output (MIMO) interference channel to three-cell MIMO cellular networks. By extending the notion of SACs to three-cell networks we show that when $d \in \mathbb{Z}^+$ DoF/user are achievable in a three-user $M \times N$ interference channel (IC) using linear beamforming, then any DoF-tuple $\{d_{ij}\}$, where $d_{ij} \in \mathbb{Z}^+$ represents the DoF of the jth user in the ith cell, that satisfies $\sum_{j=1}^{K} d_{ij} \leq d \ \forall\ i$ is achievable in a three-cell MIMO cellular network with K users per cell having M antennas per user and N antennas per base station (i.e., the $(3, K, M, N)$ network) using linear beamforming. When restricted to symmetric DoF, this result states that whenever d DoF/user are achievable in a three-user $M \times N$ IC with $d = rs$, for some $r, s \in \mathbb{Z}^+$, then r DoF/user are achievable in the $(3, s, M, N)$ network. Although the DoF achieved using SACs is not necessarily the largest possible, they are established through a constructive procedure where we show how SACs designed for the three-user interference channel can be modified to design transmit beamformers for the three-cell MIMO cellular networks. Further, we highlight the role played by redundant antennas in reducing the computational cost of designing transmit beamformers for interference alignment.**

I. SUMMARY

Degrees of freedom (DoF) is a useful metric in understanding the capacity of wireless networks and has been studied extensively [1]–[6]. While the optimal DoF of single-input single-output (SISO) wireless networks and some multiple-input multiple-output (MIMO) wireless networks [3], [4], [7] are established using the asymptotic alignment scheme of [1] or the rational dimensions framework of [2], linear beamforming schemes with finite symbol extensions play an important role in establishing the optimal DoF of several MIMO networks [8]. In particular, linear beamforming strategies for interference alignment (IA) are used to establish the optimal DoF of networks such as the three-user MIMO interference channel (IC) [8], two-cell cellular networks [9], [10] etc. In this work, we focus on the achievable DoF using linear beamforming for three-cell MIMO cellular networks. Although linear beamforming strategies for IA with finite symbol extensions are not necessarily optimal from a DoF standpoint [11], they are of interest due to their relative simplicity and ease of implementation.

Designing transmit and receive beamformers for IA is equivalent to solving a system of bilinear equations and the feasibility of such systems is studied extensively in [12]–[15]. A necessary condition for the feasibility of IA requires that the number of variables in the system of bilinear equations exceed the total number of equations. Systems satisfying this constraint are called proper systems. When d DoF/user are desired in a K-user $M \times N$ IC, the network is said to be a proper system if it satisfies $M + N \geq d(K+1)$ [12]. Similarly, when d DoF/user are desired in a MIMO cellular network with G cells, K users per cell, M antennas at each user, and N antennas at each base station (i.e., the (G, K, M, N) network), the network is said to be proper if $M + N \geq d(GK + 1)$ [13].

Computing transmit and receive beamformers that satisfy the system of bilinear equations when IA is known to be feasible is not always straightforward. Typically, iterative algorithms are used when such solutions are not readily available [13], [16]–[19]. In spite of the progress on algorithmic techniques for IA, computing the aligned transmit and receive beamformers can be computationally intensive and convergence to a set of aligned beamformers is not guaranteed.

In light of this computational overhead and an inability to guarantee convergence to aligned solutions, it is imperative to find simple non-iterative approaches that guarantee a set of aligned beamformers. This paper provides one such approach for a class of three-cell networks by extending the notion of SACs introduced in [8] for the three-user IC. A SAC identifies a sequence of transmit subspaces at different transmitters such that each transmit subspace in the sequence aligns its interference with the preceding and succeeding transmit subspaces in the chain. Once a SAC is identified, satisfying the IA conditions imposed by the chain amounts to simply solving a system of linear equations. Although the DoF achieved using this method is not necessarily the largest possible, a non-iterative construction guaranteed to yield a set of aligned transmit beamformers is valuable. Using SACs, we show that when $d \in \mathbb{Z}^+$ DoF/user are achievable in a three-user $M \times N$ IC using linear beamforming, then any DoF-tuple $\{d_{ij}\}$, where $d_{ij} \in \mathbb{Z}^+$ represents the DoF of the jth user in the ith cell, that satisfies $\sum_{j=1}^{K} d_{ij} \leq d \ \forall\ i$ is achievable in a $(3, K, M, N)$ network using linear beamforming. When restricted to symmetric DoF, this result states that whenever d DoF/user are achievable in a three-user $M \times N$ IC with $d = rs$, for some $r, s \in \mathbb{Z}^+$, then r DoF/user are achievable in a $(3, s, M, N)$ network. We also highlight the role played by redundant antennas in making it easier to compute transmit beamformers for IA.

II. Signal Model

Consider the $(3, K, M, N)$ network. Let the channel from the jth user in the ith cell to the kth base station (BS) be denoted as the $N \times M$ matrix $\mathbf{H}_{(ij,k)}$. We assume all channels to be generic. In the uplink, let $\mathbf{x}_{kl}$ denote the $M \times 1$ signal vector transmitted by the lth user in the kth cell. This transmit signal vector is formed using a $M \times d_{kl}$ linear transmit beamforming matrix $\mathbf{V}_{kl}$ and received using a $N \times d_{kl}$ receive beamforming matrix $\mathbf{U}_{kl}$, where d_{kl} represents the number of transmitted data streams of the lth user in the kth cell. The received signal after being processed by the receive beamforming matrix $\mathbf{U}_{kl}$ at the kth BS can be written as

$$\mathbf{U}_{kl}^H \mathbf{y}_k = \sum_{i=1}^{3} \sum_{j=1}^{K} \mathbf{U}_{kl}^H \mathbf{H}_{(ij,k)} \mathbf{V}_{ij} \mathbf{s}_{ij} + \mathbf{U}_{kl}^H \mathbf{n}_k. \quad (1)$$

where $\mathbf{s}_{ij}$ is the $d_{ij} \times 1$ symbol vector transmitted by the jth user in ith cell and $\mathbf{n}_k$ is the $N \times 1$ vector representing circular symmetric additive white Gaussian noise $\sim \mathcal{N}(\mathbf{0}, \mathbf{I})$. The received signal is defined similarly for the downlink. For notational clarity we use $\mathbf{W}_i$ to refer to the transmit beamformers in an IC and use $\mathbf{V}_{ij}$ when referring to transmit beamformers of a user in the uplink of a cellular network.

III. Existing Results

We first review the results on the DoF of the three-user $M \times N$ IC presented in [8]. In [8], Wang et al. consider spatial extensions of a channel where spatial dimensions are added to the system through addition of antennas at the transmitters and receivers, while keeping the ratio of antennas constant. Unlike time or frequency extensions where the resulting channels are block diagonal, spatial extensions assume generic channels with no additional structure—making them significantly easier to study without the peculiarities associated with additional structure. Using the notion of spatial extensions, spatially-normalized DoF of a (G, K, M, N) network is defined as $\text{sDoF}(M, N) = \max_{q \in \mathbb{Z}^+} \frac{\text{DoF}(qM, qN)}{q}$, where $\text{DoF}(M, N)$ is the DoF per user in a (G, K, M, N) network.

While spatial extensions may not be simple to implement physically, it has been observed that DoF achieved using spatial extensions can also be achieved using time(or frequency) extensions over time(or frequency) varying channels. The sDoF of the three-user $M \times N$ IC is completely characterized in [8] and restated in the following theorem.

Theorem 3.1 ([8]) *The spatially-normalized DoF per user of a three-user $(M \times N)$ IC is given by*

$$sDoF/user = \begin{cases} \frac{pM}{2p-1}, & \frac{p-1}{p} \le \frac{M}{N} \le \frac{2p-1}{2p+1} \\ \frac{pN}{2p+1}, & \frac{2p-1}{2p+1} \le \frac{M}{N} \le \frac{p}{p+1} \end{cases} \quad p = 1, 2, \ldots$$

The achievability is established through linear beamforming based on a notion called subspace alignment chains and only requires finite number of spatial extensions. When extensions in time/frequency/space are not allowed, the achievable DoF using this technique is given in the following theorem.

Theorem 3.2 ([8]) *The achievable DoF per user of a three-user $(M \times N)$ IC using linear beamforming without symbol extensions is given by*

$$sDoF/user = \begin{cases} \lfloor \frac{pM}{2p-1} \rfloor, & \frac{p-1}{p} \le \frac{M}{N} \le \frac{2p-1}{2p+1} \\ \lfloor \frac{pN}{2p+1} \rfloor, & \frac{2p-1}{2p+1} \le \frac{M}{N} \le \frac{p}{p+1} \end{cases} \quad p = 1, 2, \ldots$$

The basic steps involved in establishing the achievability of the result in Theorem 3.1 can be summarized as follows. Note that when $\frac{M}{N} \le \frac{1}{2}$, the optimal DoF are achieved by simple random transmit beamforming. For $\frac{1}{2} < \frac{M}{N} < 1$, except when $\frac{M}{N} = \frac{q(2p-1)}{q(2p+1)}$, either M or N has redundant dimensions, i.e., either M or N can be reduced without affecting the DoF. Further, any system with $\frac{p-1}{p} \le \frac{M}{N} \le \frac{2p-1}{2p+1}$, can be converted to a system with $M' = (2p-1)M$ and $N' = (2p+1)M$ using $(2p-1)$ space extensions and deactivating some redundant antennas. Thus, it suffices to consider systems with $\frac{M}{N} = \frac{(2p-1)}{(2p+1)}$ to establish the result in Theorem 3.1.

When $M = q(2p-1)$ and $N = q(2p+1)$ for some $p, q \in \mathbb{Z}^+$, consider transmission from nodes having M antennas to nodes having N antennas. Transmit beamformers are designed using three SACs of length p, involving q transmit beamformers in each of the p steps of a chain. As an example, when $p = 4$ and $q = 2$, we have $M = 14$ and $N = 18$ and we achieve 8 DoF/user by constructing three SACs as given below.

$$\begin{aligned} &\text{(A)} \quad \mathbf{W}_{11} \overset{Rx\ 3}{\longleftrightarrow} \mathbf{W}_{21} \overset{Rx\ 1}{\longleftrightarrow} \mathbf{W}_{31} \overset{Rx\ 2}{\longleftrightarrow} \mathbf{W}_{12} \\ &\text{(B)} \quad \mathbf{W}_{22} \overset{Rx\ 1}{\longleftrightarrow} \mathbf{W}_{32} \overset{Rx\ 2}{\longleftrightarrow} \mathbf{W}_{13} \overset{Rx\ 3}{\longleftrightarrow} \mathbf{W}_{23} \quad (2) \\ &\text{(C)} \quad \mathbf{W}_{33} \overset{Rx\ 2}{\longleftrightarrow} \mathbf{W}_{14} \overset{Rx\ 3}{\longleftrightarrow} \mathbf{W}_{24} \overset{Rx\ 1}{\longleftrightarrow} \mathbf{W}_{34}, \end{aligned}$$

where $\mathbf{W}_{ij}$ represents the jth set of q transmit beamformers corresponding to the ith transmitter. The notation $\mathbf{W}_{ij} \overset{Rx\ m}{\longleftrightarrow} \mathbf{W}_{kl}$ means the transmit beamformers $\mathbf{W}_{ij}$ and $\mathbf{W}_{kl}$ align at the mth receiver, i.e., $\mathbf{H}_{ij,m}\mathbf{W}_{ij} - \mathbf{H}_{kl,m}\mathbf{W}_{kl} = 0$. While separation between interference and signal at each of the receivers is taken care by assuming generic channels, linear independence of the pq transmit beamformers designed for each of the three transmitters is resolved through a polynomial identity test.

In general, for any given $p, q \in \mathbb{Z}^+$, the set of transmit beamformers $\mathbf{W}_{kl}$ that participate in the jth link of the ith chain can be mathematically represented using the following rule:

$$(i, j) \rightarrow \begin{cases} \mathbf{W}_{\left((i+j-1)\ \widehat{\text{mod}}\ 3,\ \lceil \frac{(i-1)p+j}{3} \rceil\right)} & \text{if } p \in \{3, 6, 9, \ldots\}, \\ \mathbf{W}_{\left((ip-p+j)\ \widehat{\text{mod}}\ 3,\ \lceil \frac{(i-1)p+j}{3} \rceil\right)} & \text{otherwise.} \end{cases} \quad (3)$$

The assignment rule ensures all transmitters are equally represented and that each of the chains originates at different transmitters. The function $a \ \widehat{\text{mod}}\ b$ returns $a \bmod b$ when a is not a multiple of b and equals b otherwise.

When symbol extensions are not allowed, the achievable DoF are established by three $\left\lfloor \frac{N}{2p+1} \right\rfloor$-dimensional SACs of length p

followed by another set of three SACs of length $p' = \left\lfloor \frac{pN}{2p+1} \right\rfloor - p \left\lfloor \frac{N}{2p+1} \right\rfloor$ involving transmit-subspaces of size one.

IV. Subspace Alignment Chains for the Three-Cell Network

Using the notion of SACs we extend the results in Theorems 3.1 and 3.2 and establish the following theorem.

Theorem 4.1 *If $d \in \mathbb{Z}^+$ DoF/user are achievable through linear beamforming over l spatial extensions in a three-user $M \times N$ IC with $M \leq N$, then any DoF-tuple $\{d_{ij}\}$, where $d_{ij} \in \mathbb{Z}^+$ represents the DoF of the jth user in the ith cell, that satisfies $\sum_{j=1}^{K} d_{ij} \leq d \; \forall \; i$ is achievable in a $(3, K, M, N)$ network using linear beamforming over l spatial extensions.*

When restricted to symmetric DoF, Theorem 4.1 can be restated as follows.

Theorem 4.2 *If $d \in \mathbb{Z}^+$ DoF/user are achievable through linear beamforming over l spatial extensions in a three-user $M \times N$ IC with $M \leq N$ and if $d = rs$ for some $r, s \in \mathbb{Z}^+$, then r DoF/user are achievable through linear beamforming in a $(3, s, M, N)$ cellular network over l spatial extensions.*

Since the proof techniques and the key insights in establishing Theorems 4.1 and 4.2 are the same, we restrict ourselves to the symmetric DoF case in the rest of the paper.

Focusing on the result of Theorem 4.2, note that when rs DoF/user are achievable in a three-user IC without any spatial extensions, r DoF/user in a three-cell, s users/cell network can also be achieved by scheduling one user per cell per time slot to form a three-user MIMO IC and then cycling through the s users in each cell over s time slots. While such a strategy requires s time slots to achieve r DoF/user, Theorem 4.2 suggests that the same number of DoF are achievable within one time slot. The DoF that can be achieved simultaneously without time sharing has important consequences in user scheduling in cooperative cellular networks. Since DoF are the number of interference free directions available in a network, the achievable degrees of freedom inform us of the right number of users to simultaneously schedule and the number of data streams to deliver per user.

As a simple illustration of Theorem 4.2, consider the three-user 3×5 IC for which 2 DoF/user are achievable without any symbol extensions. Theorem 4.2 states that since 2 DoF/user are known to be feasible for such a network without any symbol extensions, we can achieve 1 DoF/user for the $(G = 3, K = 2, M = 3, N = 5)$ cellular network without any symbol extensions. While sufficient conditions in [15] also show such a system to be feasible, a key advantage to our approach is that we establish feasibility by constructing the necessary transmit beamformers using SACs.

V. Proof of Theorem 4.2

The proof of Theorem 4.2 relies on reusing the SACs designed for the three-user IC with a careful redistribution of the transmit-subspaces that participate at each stage of the chain among the multiple users in a three-cell network. For brevity, we only consider the case when sufficient spatial extensions are allowed so that we only need to construct three SACs to achieve the requisite number of DoF; the same arguments can be applied in a straightforward manner to the case when DoF are achieved without any spatial extensions. We consider linear beamforming in the uplink and design transmit beamformers for each user in the network.

When $\frac{M}{N} \leq \frac{1}{2}$, the result is straightforward to establish using zero-forcing beamforming. The cases when $\frac{1}{2} < \frac{M}{N} < 1$ and $\frac{M}{N} = 1$ require the construction of two types of SACs and are discussed separately.

Case (i): $\frac{1}{2} < \frac{M}{N} < 1$: When $\frac{1}{2} < \frac{M}{N} < 1$, it suffices to consider the cases when $M = (2p-1)q$ and $N = (2p+1)q$. We assume $pq = rs$ and show that if pq DoF are achievable in a three-user $(2p-1)q \times (2p+1)q$ IC then r DoF/user are achievable in a $(3, s, (2p-1)q, 2p+1)q)$ cellular network.

For the three-user IC, once a sequence of transmit subspaces that participate in a SAC are chosen, this also identifies a sequence of receivers at which these subspaces align. For example, for the subspace chain A in (2) the sequence of receivers is 3-1-2. A crucial insight in extending the notion of SACs to three-cell networks is that the feasibility of an alignment chain is unaltered as long as the sequence of receivers (BSs) of the original chain is preserved even if the transmit-subspaces that participate in the alignment chain now belong to multiple users within a cell.

When $\frac{1}{2} < \frac{M}{N} < 1$, note that the three SACs in the case of the three-user $M \times N$ IC can be viewed as $3q$ vector alignment chains with each chain comprising a sequence of p transmit beamforming vectors that align with the preceding and succeeding vectors in the chain. To illustrate this, the three SACs in (2) involving two beamformers at each stage can be viewed as the following six vector alignment chains:

$$\begin{array}{ll}
\text{(1A)} & \mathbf{w}_{11} \xleftrightarrow{Rx\ 3} \mathbf{w}_{21} \xleftrightarrow{Rx\ 1} \mathbf{w}_{31} \xleftrightarrow{Rx\ 2} \mathbf{w}_{12} \\
\text{(1B)} & \mathbf{w}_{22} \xleftrightarrow{Rx\ 1} \mathbf{w}_{32} \xleftrightarrow{Rx\ 2} \mathbf{w}_{13} \xleftrightarrow{Rx\ 3} \mathbf{w}_{23} \\
\text{(1C)} & \mathbf{w}_{33} \xleftrightarrow{Rx\ 2} \mathbf{w}_{14} \xleftrightarrow{Rx\ 3} \mathbf{w}_{24} \xleftrightarrow{Rx\ 1} \mathbf{w}_{34} \\
\text{(2A)} & \mathbf{w}_{15} \xleftrightarrow{Rx\ 3} \mathbf{w}_{25} \xleftrightarrow{Rx\ 1} \mathbf{w}_{35} \xleftrightarrow{Rx\ 2} \mathbf{w}_{16} \\
\text{(2B)} & \mathbf{w}_{26} \xleftrightarrow{Rx\ 1} \mathbf{w}_{36} \xleftrightarrow{Rx\ 2} \mathbf{w}_{17} \xleftrightarrow{Rx\ 3} \mathbf{w}_{27} \\
\text{(2C)} & \mathbf{w}_{37} \xleftrightarrow{Rx\ 2} \mathbf{w}_{18} \xleftrightarrow{Rx\ 3} \mathbf{w}_{28} \xleftrightarrow{Rx\ 1} \mathbf{w}_{38}.
\end{array} \tag{4}$$

Denoting the original 3 chains as A, B and C, let the $3q$ chains be labeled 1A, 2A, ..., qA, 1B, ..., qB, 1C, ..., qC, as shown in (2) and (4). We lexicographically order these $3q$ chains and refer to them using indices from 1 to $3q$. Among these $3q$ vector alignment chains for the three-user IC, there are pq instances where transmit beamformers from the kth transmitter are chosen. We reassign these pq instances to transmit beamformers corresponding to the s users in the kth cell using the rule in (5).

Since the first subscript in the reassignment rule (3) is identical to that in (5), this rule preserves the original receiver (now BSs) sequence, thus ensuring feasibility of the new alignment chain. The second subscript picks users in a cyclic manner, ensuring equal distribution of the transmit-subspaces among the different users in a cell. The third subscript keeps count of the beamformer index. In (6) we illustrate an example of

$$(i,j) \rightarrow \begin{cases} \mathbf{v}_{\left((i+j-1)\, \widehat{\bmod}\, 3,\ \lceil \frac{ip-p+j}{3} \rceil\, \widehat{\bmod}\, s,\ \lceil \frac{ip-p+j}{3s} \rceil \right)} & \text{if } p \in \{3,6,9,\ldots\}, \\ \mathbf{v}_{\left((ip-p+j)\, \widehat{\bmod}\, 3,\ \lceil \frac{ip-p+j}{3} \rceil\, \widehat{\bmod}\, s,\ \lceil \frac{ip-p+j}{3s} \rceil \right)} & \text{otherwise.} \end{cases} \tag{5}$$

constructing a set of SACs for the $(3,4,14,18)$ cellular network where 2 DoF/user are achieved without symbol extensions. The new set of chains are derived directly from the corresponding chains for the 3-user 14×18 IC shown in (4) and can be expressed as follows

$$\begin{array}{ll} \text{(1A)} & \mathbf{v}_{111} \xleftrightarrow{BS\ 3} \mathbf{v}_{211} \xleftrightarrow{BS\ 1} \mathbf{v}_{311} \xleftrightarrow{BS\ 2} \mathbf{v}_{121} \\ \text{(1B)} & \mathbf{v}_{221} \xleftrightarrow{BS\ 1} \mathbf{v}_{321} \xleftrightarrow{BS\ 2} \mathbf{v}_{131} \xleftrightarrow{BS\ 3} \mathbf{v}_{231} \\ \text{(1C)} & \mathbf{v}_{331} \xleftrightarrow{BS\ 2} \mathbf{v}_{141} \xleftrightarrow{BS\ 3} \mathbf{v}_{241} \xleftrightarrow{BS\ 1} \mathbf{v}_{341} \\ \text{(2A)} & \mathbf{v}_{112} \xleftrightarrow{BS\ 3} \mathbf{v}_{212} \xleftrightarrow{BS\ 1} \mathbf{v}_{312} \xleftrightarrow{BS\ 2} \mathbf{v}_{122} \\ \text{(2B)} & \mathbf{v}_{222} \xleftrightarrow{BS\ 1} \mathbf{v}_{322} \xleftrightarrow{BS\ 2} \mathbf{v}_{132} \xleftrightarrow{BS\ 3} \mathbf{v}_{232} \\ \text{(2C)} & \mathbf{v}_{332} \xleftrightarrow{BS\ 2} \mathbf{v}_{142} \xleftrightarrow{BS\ 3} \mathbf{v}_{242} \xleftrightarrow{BS\ 1} \mathbf{v}_{342}. \end{array} \tag{6}$$

Once the conditions for IA are expressed in terms of vector alignment chains, the aligned transmit beamformers are computed by solving a system of linear equations that emerge from the alignment chains. While SACs ensure that interference is contained within a certain number of dimensions, the assumption of generic channels and the fact that direct channels are not involved in the design of the transmit beamformers ensure that signal and interference are separable at each of the three BSs. The only remaining issue concerns linear independence of the transmit beamformers designed for any particular user. This issue can be resolved through a polynomial identity test similar to the one used in Section 8.1 of [8].

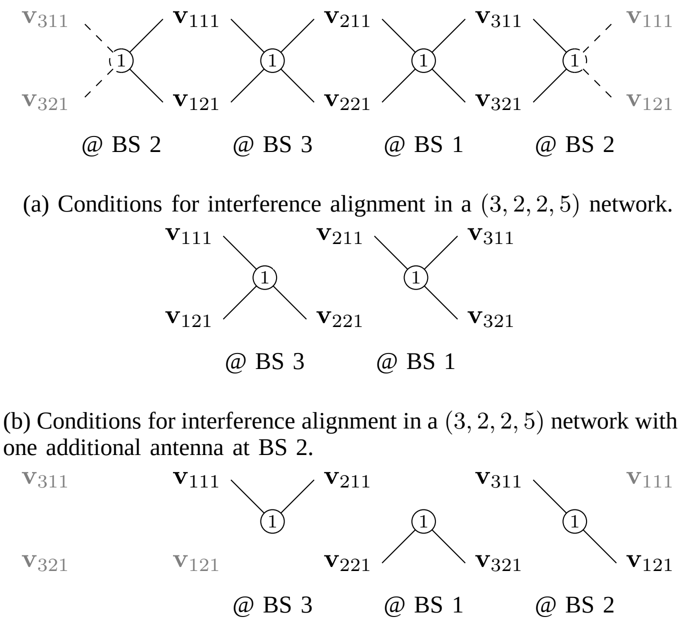
Case (ii):$\frac{M}{N} = 1$: When $M = N$, the transmit beamformers can be designed by identifying them as the eigenvectors of an effective channel matrix, similar to the method proposed in [1]. In particular, suppose $M = 2p$ and $N = 2p$ for some $p \in \mathbb{Z}^+$, then p DoF/user are achievable in a 3-user IC. If $p = rs$ for some $r, s \in \mathbb{Z}^+$, then r DoF/user can be achieved in a $(3, s, 2p, 2p)$ cellular network by solving the following closed-loop SAC for all $j \in \{1,2,\ldots,s\}$ and $i \in \{1,2,\ldots,r\}$. Exact details on solving this alignment chain can be found in [1].

$$\begin{array}{ccc} & \mathbf{v}_{2ji} & \\ {}^{BS\ 3}\nearrow\!\swarrow & & \nwarrow\!\searrow^{BS\ 1} \\ \mathbf{v}_{1ji} & \xleftrightarrow{BS\ 2} & \mathbf{v}_{3ji} \end{array} \tag{7}$$

VI. Role of Redundant Antennas

While SACs provide a simple way to design aligned beamformers for three-cell MIMO cellular networks, this comes at the cost of requiring some redundancy in the number of antennas at the users or the BSs so as to meet a certain DoF target. As a specific example to highlight this point, using the conditions that a feasible system needs to be proper, it can be shown that when we desire 1 DoF/user for a $(G = 3, K, M, N)$ cellular network, we need to ensure $M + N \geq 3K + 1$ (when $d = 1$, proper systems with $N \geq K$ are feasible [15]). However, if we were to use the subspace alignment technique provided by Theorem 4.2, we need to at least ensure that $M + N \geq 4K$, suggesting that we need at least $(K-1)$ additional antennas at each BS/user to achieve the same DoF target. This illustrates the trade off between redundant antennas and the computational ease of finding aligned beamformers. To further illustrate the role of redundant antennas, consider Fig. 1(a), where we have graphically represented the conditions necessary to achieve 1 DoF/user in a $(3,2,2,5)$ cellular network in the uplink. In Fig. 1, every circle represents a set of linear equations that need to be satisfied by the transmit beamformers that are connected to it. For example, since BS 1 has 5 antennas, we require the four interfering streams from users in cells 2 and 3 to occupy no more than 3 dimensions at BS 1. This requires the corresponding transmit beamformers to satisfy a linear equation of the form

$$\alpha^1_{211}\mathbf{H}_{21,1}\mathbf{v}_{211} + \alpha^1_{221}\mathbf{H}_{22,1}\mathbf{v}_{221} + \alpha_{311}\mathbf{H}_{31,1}\mathbf{v}_{311} + \alpha_{321}\mathbf{H}_{32,1}\mathbf{v}_{321} = \mathbf{0}. \tag{8}$$

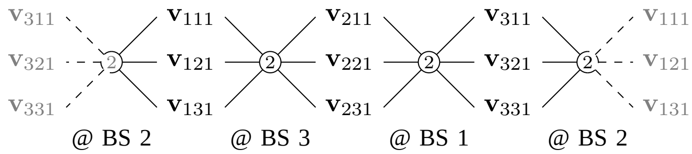


(a) Conditions for interference alignment in a $(3,2,2,5)$ network.

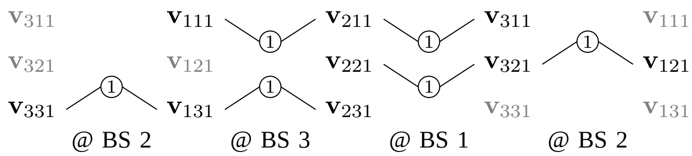
(b) Conditions for interference alignment in a $(3,2,2,5)$ network with one additional antenna at BS 2.

(c) Conditions for interference alignment in a $(3,2,3,5)$ network.

Fig. 1: Conditions for IA (in the uplink) to achieve 1 DoF/user in three different networks. Each circle represents a set of linear equations to be satisfied by the transmit beamformers connected to it at a certain BS (indicated below). The number inside the circle represents the number of linear equations to be satisfied. Note that the transmit beamformers at either end (in gray) are duplicates, indicating a closed-loop.



(a) Conditions for interference alignment in a (3,3,3,7) network.



(b) Conditions for interference alignment in a (3,3,5,7) network.

Fig. 2: Conditions for IA to achieve 1 DoF/user in two three-cell networks with three users/cell.

where $\{\alpha_{ijk}^l\}$ represents a set of coefficients. Note that the conditions for alignment represented in Fig. 1(a) form a closed-loop, resulting in a tightly coupled system of equations. In addition to the tight coupling, since the coefficients defining the linear equations are not known a priori, this set of equations is not easy to solve. Treating the coefficients as additional unknowns to be determined, we obtain a system of bilinear equations that are much harder to solve than just linear equations.

In such a situation, adding redundant antennas at the users or the BSs helps in (a) decoupling the linear equations and (b) determining the coefficients associated with these equations. For example, suppose we add one additional antenna to BS 2 of the $(3,2,2,5)$ network represented in Fig. 1(a), alignment is no longer necessary at BS 2 and further, the alignment conditions at BSs 1 and 3 can be decoupled as shown in Fig. 1(b). Having decoupled the equations, we can now set all coefficients to one and solve for the transmit beamformers.

Alternately, if we add an additional antenna to each of the users, then the alignment conditions can be decoupled as shown in Fig. 1(c). Note that with three antennas at each user, it suffices to consider just two transmit beamformers for each alignment condition, resulting in three alignment chains of length two, exactly similar to the construction suggested in Section V. The coefficients in this case are all set to one, and it is straightforward to solve for the transmit beamformers.

As another example, consider designing linear beamformers to achieve 1 DoF/user in a $(3,3,3,7)$ cellular network. The six interfering streams at each BS now need to satisfy two linear equations as shown in Fig. 2(a). By adding two antennas to each user, it suffices to consider just two transmit beamformers for each alignment condition, thus forming three alignment chains of length three as shown in Fig. 2(b).

It is evident from the examples provided in this section that while SACs are convenient from a computational perspective, they fall short of achieving the optimal DoF that can be achieved using linear beamforming. Since such chains restrict the structure of IA by allowing only two transmit subspaces for each linear equation of the form (8), they do not take advantage of other avenues for alignment that now become possible due to the presence of multiple users per cell. Fig. 1(b) illustrates one such type of alignment that is not captured by the basic template for alignment provided by alignment chains. Structures that exploit these new avenues for alignment are necessary to establish the optimal DoF in regimes where linear beamforming schemes outperform asymptotic alignment schemes.

VII. Conclusion

In this paper we extend the notion of SACs to three-cell networks. We provide a constructive approach to designing transmit beamformers for certain three-cell networks by modifying the SACs designed for a corresponding three-user IC. We highlight the inherent trade-off in the ease of finding such aligned beamformers and the redundancy in the number of antennas at each transmitter/receiver in the network and provide insights on the role played by redundant antennas.

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