# Gaussian Z-Interference Channel with a Relay Link: Achievability Region and Asymptotic Sum Capacity

Wei Yu\*, Senior Member, IEEE and Lei Zhou, Student Member, IEEE

Abstract—This paper studies a Gaussian Z-interference channel with a rate-limited digital relay link from one receiver to another. Achievable rate regions are derived based on a combination of Han-Kobayashi common-private information splitting technique and several different relay strategies including decodeand-forward, quantize-and-forward, and a partial-interferenceforwarding strategy, in which the interference is decoded then binned and forwarded through the digital link for interference subtraction at the other end. For the Gaussian Z-interference channel with a digital link from the interference-free receiver to the interfered receiver, the capacity region is established in the strong interference regime; an achievable rate region is established in the weak interference regime. In addition, in the weak interference regime, the partial-interference-forwarding strategy is shown to be asymptotically sum-capacity achieving in the high signal-to-noise ratio and high interference-to-noise ratio limit. In this case, each relay bit asymptotically improves the sum capacity by one bit. For the Gaussian Z-interference channel with a digital link from the interference-free receiver to the interfered receiver, the capacity region is established in the strong interference regime; achievable rate regions are established in the moderately strong and weak interference regime. In the weak interference regime, the quantize-and-forward strategy is shown to be asymptotically sum-capacity achieving in the limit of large relay link rate. However, in this case, the sum capacity improvement due to the digital link is shown to be bounded by half a bit.

#### I. INTRODUCTION

The classic interference channel models a communication situation in which two transmitters communicate with their respective intended receivers while mutually interfering with each other. The interference channel is of fundamental importance for communication system design, because many practical systems are designed to operate in the interferencelimited regime. The largest known achievability region for the interference channel is due to Han and Kobayashi [1], where a common-private information splitting technique is used to partially decode and subtract the interfering signal. The Han-Kobayashi scheme has recently been shown to be capacity achieving in a very weak interference regime [2], [3], [4] and to be within one bit of the capacity region in general [5].

This paper considers a novel communication model in which the classic interference channel is augmented by a noiseless relay link between the two receivers. We are motivated to study such a relay interference channel because in many practical communication situations (e.g. a wireless cellular system with remote users at the cell edge), the receivers are often close to each other geographically and are capable of establishing an independent communication link for the purpose of interference mitigation.

This paper explores the use of relay techniques for interference mitigation. We focus on the simplest interference channel model, the Gaussian Z-interference channel (also known as the one-sided interference channel), in which one of the receivers gets an interference-free signal, the other receiver gets a combination of the intended and the interfering signals, and the channel is equipped with a noiseless link of fixed capacity from one receiver to the other. Depending on the direction of the noiseless link, the proposed model is named the Type I or the Type II Gaussian Z-relay-interference channel, as shown in Fig. 1.

The Type I Gaussian Z-relay-interference channel has a digital relay link of finite capacity from the interference-free receiver to the interfered receiver. Our main coding strategy for the Type I channel is a relay scheme called partial interference forwarding, in which the relay link is used for partial interference subtraction at the interfered receiver. The idea is to explore the fact that interference consists of structured codewords, which can be described efficiently using a binning technique. This extra piece of binning information enhances the interference subtraction capability at the interfered receiver, thereby enlarging the overall achievable region for the interference channel. Interestingly, our analysis shows that even though the relay link is from the interference-free receiver to the interfered receiver, it can benefit the interference-free receiver more than it can for the interfered receiver.

This paper also shows that partial interference forwarding is capacity achieving for the Type I channel in the strong interference regime, and is asymptotically sum-capacity achieving in the weak interference regime. In fact, in the weak interference regime, every bit of relay link rate increases the sum rate by one bit in the high signal-to-noise ratio (SNR) and high interference-to-noise ratio (INR) limit. Essentially, a digital

Manuscript submitted to the *IEEE Transactions on Information Theory* on September 3, 2008. The materials in this paper will be presented in part at the 2008 International Symposium on Information Theory and its Applications (ISITA), December 2008, in Auckland, New Zealand. Wei Yu is with the Electrical and Computer Engineering Department, University of Toronto, 10 King's College Road, Toronto, Ontario M5S 3G4, Canada. email: weiyu@comm.utoronto.ca. Phone: 416-946-8665. FAX: 416-978-4425. Lei Zhou was with the Electrical and Computer Engineering Department, University of Toronto. He is now with Nortel Networks, 3500 Carling Avenue, Napean, ON, Canada. email: leizh3@nortel.com. This work was supported in part by the Natural Science and Engineering Research Council (NSERC) of Canada under the Canada Research Chairs program, in part by LG Electronics, and in part by the Ontario Early Researcher Awards program. Kindly address correspondence to Wei Yu (weiyu@comm.utoronto.ca).



Fig. 1. Gaussian Z-interference channel with a relay link: (a) Type I; (b) Type II.

relay from the interference-free receiver to the interfered receiver asymptotically achieves the cut-set bound for sum capacity.

The Type II Gaussian Z-relay-interference channel differs from the Type I channel in that the direction of the digital link now goes from the interfered receiver to the interference-free receiver. Our main coding strategy for the Type II channel is based on a combination of the Han-Kobayashi commonprivate information splitting scheme and two relaying strategies: decode-and-forward and quantize-and-forward. In the proposed scheme, interfered receiver, which decodes the common message and observes a noisy version of the neighbor's private message, describes the common message with a bin index and describes the neighbor's private message using a quantization scheme. It is shown that, in the strong interference regime, a special form of the proposed relaying scheme, which deploys the decode-and-forward strategy only, is capacity achieving. On the other hand, in the weak interference regime, the largest rate region is achieved with pure quantize-andforward. Further, in the weak interference regime, the sum capacity gain due to the digital link for the Type II channel is upper bounded by half a bit. This is in direct contrast to the Type I channel, in which each relay bit can be worth up to one bit in sum capacity.

#### A. Literature Review

The Gaussian Z-interference channel has been extensively studied in the literature. It is one of the few examples of an interference channel (besides the strong interference case [6], [7], [1] and the very weak interference case [2], [3], [4]) for which the sum capacity has been established. The sum capacity for the Gaussian Z-interference channel in the weak interference regime is achieved with both transmitters using Gaussian codebooks and with the interfered receiver treating the interference as noise [8], [5].

The fundamental decode-and-forward and quantize-andforward strategies for the relay channel are due to the classic work of Cover and El Gamal [9]. Our study of the interference channel with a relay link is motivated by the more recent capacity results for a class of deterministic relay channels investigated by Kim [10] and a class of modulo-sum relay channels investigated by Aleksic et al. [11], where the relay observes the noise in the direct channel. The situation



investigated in [10], [11] is similar to the Type I Gaussian Zrelay-interference channel, where the interference-free receiver observes a noisy version of the interference at the interfered receiver and helps the interfered receiver by describing the interference through the noiseless relay link.

The channel model studied in the paper is related to the recent work of Sahin and Erkip [12], [13], Marić et al. [14] and Dabora et al. [15], where the achievable rate regions and relay strategies are studied for an interference channel with an additional relay node. The relay observes the transmitted signals from the inputs and contributes to the outputs of both channels. In particular, [14], [15] propose an interference-forwarding strategy which is similar to the one used for Type I channel in this paper. However, the channel model of this paper is considerably simpler. By focusing on a Gaussian Z-interference channel with a "primitive" relay (in the sense of [16]), we are able to derive more concrete achievability results and upper bounds.

## B. Outline of the Paper

The rest of this paper is organized as follows. Section II presents achievability results for the Type I Gaussian Z-relayinterference channel, where a partial-interference-forwarding strategy is used. Capacity results are established for the strong interference case. Asymptotic sum capacity result is established for the weak interference case in the high SNR/INR limit. Section III presents achievability results for the Type II Gaussian Z-relay-interference channel using a combination of the decode-and-forward scheme and the quantize-and-forward scheme. Capacity results are derived in the strong interference regime; capacity upper bound is derived in the weak interference regime. Section IV provides concluding remarks and a generalization of our results to the general Gaussian interference channel with a relay link.

## II. GAUSSIAN Z-INTERFERENCE CHANNEL WITH A RELAY LINK: TYPE I

#### A. Channel Model

The Gaussian Z-interference channel is modeled as follows (see Fig. 1(a)):

$$\begin{cases} Y_1 = h_{11}X_1 + h_{21}X_2 + Z_1 \\ Y_2 = h_{22}X_2 + Z_2 \end{cases}$$
(1)

where  $X_1$  and  $X_2$  are the transmit signals with power constraints  $P_1$  and  $P_2$  respectively,  $h_{ij}$  represents the channel gain from transmitter *i* to receiver *j*, and  $Z_1$ ,  $Z_2$  are independent additive white Gaussian noises (AWGN) with power *N*. In addition, the Type I Gaussian Z-relay-interference channel is equipped with a digital noiseless link of fixed capacity  $R_0$ from the receiver 2 to the receiver 1.

To simplify the notation, the following definitions are used throughout this paper:

$$SNR_{1} = \frac{|h_{11}|^{2}P_{1}}{N} \qquad SNR_{2} = \frac{|h_{22}|^{2}P_{2}}{N}$$
$$INR_{2} = \frac{|h_{21}|^{2}P_{2}}{N} \qquad \gamma(x) = \frac{1}{2}\log(1+x)$$

where  $\log(\cdot)$  is base 2. In addition, denote  $\overline{\beta} = 1 - \beta$ .

#### B. Achievable Rate Region

Given  $Y_2$ 's observation, how should it utilize the noiseless relay link to help  $Y_1$  decode  $X_1$ ? Clearly, decode-and-forward is not useful, as  $X_1$  is not observed at  $Y_2$ . Instead,  $Y_2$  observes a noisy version of the interference at  $Y_1$ . Thus, quantize-andforward may be a sensible strategy in which  $Y_2$  is described to  $Y_1$  using a rate-constrained quantization codebook. However, quantize-and-forward does not take into account the fact that  $X_2$  is not a Gaussian random noise, but a codeword from a structured codebook. In addition, one may intentionally design  $X_2$  in order to facilitate interference subtraction. This extra degree of freedom allows us to achieve a higher rate than the rate achievable with quantize-and-forward.

This paper proposes a combination of the Han-Kobayashi common-private information splitting and a bin-and-forward strategy for the Gaussian Z-relay-interference channel, in which a common information stream is intentionally designed for decoding at  $Y_2$  and forwarding to  $Y_1$  for subtraction. We call such a relay strategy a *partial-interference-forwarding* strategy. Our main result is the following achievability theorem.

Theorem 1: For the Type I Gaussian Z-interference channel with a digital relay link of limited rate  $R_0$  from the interference-free receiver to the interfered receiver as shown in Fig. 1(a), in the weak interference regime defined by  $INR_2 \leq SNR_2$ , the following rate region is achievable:

$$\bigcup_{0 \le \beta \le 1} \left\{ (R_1, R_2) \left| R_1 \le \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right), \\ R_2 \le \min \left\{ \gamma(\mathsf{SNR}_2), \gamma(\beta \mathsf{SNR}_2) + \right. \\ \left. \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + R_0 \right\} \right\}. \quad (2)$$

In the strong interference regime, defined by

$$\mathsf{SNR}_2 \le \mathsf{INR}_2 \le \max\{\mathsf{SNR}_2, \mathsf{INR}_2^*\},\tag{3}$$

where

$$\mathsf{INR}_2^* = (1 + \mathsf{SNR}_1)(2^{-2R_0}(1 + \mathsf{SNR}_2) - 1), \qquad (4)$$



Fig. 2. Common-private information splitting for Type I channel.

the capacity region is given by

$$\left\{ (R_1, R_2) \middle| \begin{array}{rcl}
R_1 &\leq & \gamma(\mathsf{SNR}_1) \\
R_2 &\leq & \gamma(\mathsf{SNR}_2) \\
R_1 + R_2 &\leq & \gamma(\mathsf{SNR}_1 + \mathsf{INR}_2) + R_0 \end{array} \right\}.$$
(5)

In the very strong interference regime defined by

$$\mathsf{NR}_2 \ge \max\{\mathsf{SNR}_2, \mathsf{INR}_2^*\},\tag{6}$$

the capacity region is given by

$$\left\{ (R_1, R_2) \middle| \begin{array}{c} R_1 \le \gamma(\mathsf{SNR}_1) \\ R_2 \le \gamma(\mathsf{SNR}_2) \end{array} \right\}.$$
(7)

**Proof:** We use the Han-Kobayashi [1] common-private information splitting scheme with Gaussian input to prove the achievability of the rate regions (2), (5) and (7). The encoding procedure is as depicted in Fig. 2. User 1's signal  $X_1$  is intended for decoding at  $Y_1$  only. User 2's signal  $X_2$  is the superposition of private message  $U_2$  and common message  $W_2$ . The private message can only be decoded by the intended receiver  $Y_2$ , while the common message can be decoded by both receivers. Independent Gaussian codebooks of sizes  $2^{nS_1}$ ,  $2^{nS_2}$  and  $2^{nT_2}$  are generated according to i.i.d. Gaussian distributions  $X_1 \sim \mathcal{N}(0, P_1), U_2 \sim \mathcal{N}(0, \beta P_2)$ , and  $W_2 \sim \mathcal{N}(0, \overline{\beta} P_2)$ , respectively, where  $0 \le \beta \le 1$ .

Decoding takes place in two steps. First,  $(W_2, U_2)$  are decoded at  $Y_2$ . The set of achievable rates  $(T_2, S_2)$  is the capacity region of a Gaussian multiple-access channel, denoted here by  $C_2$ , where

$$\begin{cases} T_2 \leq \gamma(\overline{\beta}\mathsf{SNR}_2)\\ S_2 \leq \gamma(\beta\mathsf{SNR}_2)\\ S_2 + T_2 \leq \gamma(\mathsf{SNR}_2). \end{cases}$$
(8)

After  $(W_2, U_2)$  are decoded at  $Y_2$ ,  $(X_1, W_2)$  are then decoded at  $Y_1$  with  $U_2$  treated as noise, but with the help of the relay link. This is a multiple-access channel with a rate-limited relay  $Y_2$ , who has complete knowledge of  $W_2$ . Denote the capacity region of such a multiple-access channel with a digital relay link  $R_0$  by  $C_1$ . We prove in Lemma 1 in Appendix A that a bin-and-forward relay strategy is capacity achieving in this case. The capacity region  $C_1$  is the set of  $(S_1, T_2)$  for which

$$\begin{cases} S_{1} \leq \gamma \left( \frac{\mathsf{SNR}_{1}}{1 + \beta \mathsf{INR}_{2}} \right) \\ T_{2} \leq \gamma \left( \frac{\overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}} \right) + R_{0} \\ S_{1} + T_{2} \leq \gamma \left( \frac{\mathsf{SNR}_{1} + \overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}} \right) + R_{0}. \end{cases}$$
(9)



Fig. 3. Incorporating  $C_1$  and  $C_2$  for Type I channel.



Fig. 4. Pentagon achievable rate region.

An achievable rate region of the Gaussian Z-interference channel with a relay link is then the set of all  $(R_1, R_2)$  such that  $R_1 = S_1$  and  $R_2 = S_2 + T_2$  for some  $(S_1, T_2) \in C_1$  and  $(S_2, T_2) \in C_2$ . Further, since  $C_1$  and  $C_2$  depend on the privatecommon power splitting ratio  $\beta$ , the convex hull of the union of all such  $(R_1, R_2)$  sets over all choices of  $\beta$  is achievable.

In the following, we first show that for each fixed  $\beta$ , the set of achievable rates  $(R_1, R_2)$  is a pentagon. We then show that the convex hull of the union of these pentagons reduces to the regions (2), (5) and (7).

Fig. 3 illustrates an example of a  $(S_2, T_2)$ -pentagon (8) and a  $(S_1, T_2)$ -pentagon (9) in a three dimensional diagram. As  $S_1$  moves from its maximal value downward (i.e. from A to C), the maximal achievable  $T_2 + S_2$  increases from the sum rate corresponding to A' to that corresponding to C'. The sum rate  $T_2 + S_2$  is a constant beyond C', because the line segment C'D' is at 45 degrees. Now, since the line segment AC is also at 45 degrees, from A to C, the sum  $S_1 + T_2$  is a constant as well. Consequently  $S_1 + T_2 + S_2$  is also a constant from A to C. Therefore, when plotting  $R_2$  (which is  $S_2 + T_2$ ) vs.  $R_1$ (which is  $S_1$ ), we obtain a pentagon shape with a 45-degree edge A''C'' as shown in Fig. 4. Beyond point C,  $S_2+T_2$  stays as a constant as  $S_1$  decreases.

Mathematically, it is possible to use a Fourier-Motzkin elimination (see e.g. [17]) to verify that for each fixed  $\beta$ , the achievable  $(R_1, R_2)$ 's form a pentagon region characterized



Fig. 5. The union of rate region pentagons in the weak interference regime.

by

$$\mathcal{R}_{\beta} = \begin{cases} \left(R_{1}, R_{2}\right) & R_{1} \leq \gamma \left(\frac{\mathsf{SNR}_{1}}{1 + \beta \mathsf{INR}_{2}}\right) \\ R_{2} \leq \min\{\gamma(\mathsf{SNR}_{2}), \gamma(\beta \mathsf{SNR}_{2}) + \\ \gamma \left(\frac{\overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}}\right) + R_{0} \right) \\ R_{1} + R_{2} \leq \gamma(\beta \mathsf{SNR}_{2}) + \\ \gamma \left(\frac{\mathsf{SNR}_{1} + \overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}}\right) + R_{0} \\ \end{cases}$$
(10)

The convex hull of the union of these pentagons over  $\beta$  gives the complete achievability region.

It turns out that the union of the pentagons, i.e.  $\bigcup_{0 \le \beta \le 1} \mathcal{R}_{\beta}$ , is a convex region. Therefore, convex hull is not needed here. In the following, we give a proof of this fact and give an explicit expression for the resulting achievable region.

Consider first the weak interference regime, where  $INR_2 \leq SNR_2$ . Ignore for now the constraint  $R_2 \leq \gamma(SNR_2)$  and focus on an expanded pentagon defined by  $\{(R_1, R_2) | R_1 \leq f_1(\beta), R_2 \leq f_2(\beta), R_1 + R_2 \leq f_3(\beta)\}$ , where  $f_1(\beta), f_2(\beta)$  and  $f_2(\beta)$  are defined as

$$f_1(\beta) = \gamma \left(\frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2}\right) \tag{11}$$

$$f_2(\beta) = \gamma \left(\beta \mathsf{SNR}_2\right) + \gamma \left(\frac{\beta \mathsf{INR}_2}{1 + \beta \mathsf{INR}_2}\right) + R_0 \qquad (12)$$

$$f_{3}(\beta) = \gamma \left(\beta \mathsf{SNR}_{2}\right) + \gamma \left(\frac{\mathsf{SNR}_{1} + \overline{\beta}\mathsf{INR}_{2}}{1 + \beta\mathsf{INR}_{2}}\right) + R_{0}(13)$$

It is easy to verify that when  $\beta = 1$ , the expanded pentagon reduces to a rectangular region, as shown in Fig. 5. Further, as  $\beta$  decreases from 1 to 0,  $f_1(\beta)$  is monotonically increasing and both  $f_2(\beta)$  and  $f_3(\beta)$  are monotonically decreasing, while  $f_2(\beta) - f_3(\beta)$  remains a constant in the INR<sub>2</sub>  $\leq$  SNR<sub>2</sub> regime. Since  $f_1(\beta)$ ,  $f_2(\beta)$  and  $f_3(\beta)$  are all continuous functions of  $\beta$ , as  $\beta$  decreases from 1 to 0, the upper left corner point of the expanded pentagon moves vertically downward in the  $R_2 - R_1$ plane, while the lower right corner point moves downward and to the right in a continuous fashion. Consequently, the union of these expanded pentagons is defined by  $R_1 \leq \gamma(SNR_1)$ ,  $R_2 \leq \gamma(SNR_2) + R_0$ , and lower-right corner points of the



Fig. 6. The union of rate region pentagons in the strong interference regime.

pentagons  $(R_1, R_2)$  with

$$\begin{cases} R_1 = \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right) \\ R_2 = \gamma (\beta \mathsf{SNR}_2) + \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + R_0 \end{cases}$$
(14)

where  $0 \le \beta \le 1$ . We prove in Appendix B that such a region is convex when  $INR_2 \le SNR_2$ . Thus, convex hull is not needed. Finally, incorporating the constraint  $R_2 \le \gamma(SNR_2)$  gives the weak-interference achievable region (2).

Now, consider the strong interference regime, where  $INR_2 \ge SNR_2$ . In this regime,  $f_1(\beta)$ ,  $f_2(\beta)$  and  $f_3(\beta)$  are all increasing functions as  $\beta$  goes from 1 to 0. Consequently,  $\bigcup_{0 \le \beta \le 1} \mathcal{R}_{\beta} = \mathcal{R}_0$ , as illustrated in Fig. 6. Thus, the achievable rate region simplifies to

$$\left\{ \left( R_1, R_2 \right) \middle| \begin{array}{l} R_1 \leq \gamma(\mathsf{SNR}_1) \\ R_2 \leq \min\{\gamma(\mathsf{SNR}_2), \gamma(\mathsf{INR}_2) + R_0\} \\ R_1 + R_2 \leq \gamma(\mathsf{SNR}_1 + \mathsf{INR}_2) + R_0 \end{array} \right\}$$
(15)

which is equivalent to (5) by noting that

$$\gamma(\mathsf{INR}_2) + R_0 \ge \gamma(\mathsf{SNR}_2) \tag{16}$$

when  $INR_2 \ge SNR_2$ .

Finally, in the very strong interference regime with  $INR_2 \ge \max{SNR_2, INR_2^*}$ , where  $INR_2^* = (1 + SNR_1)(2^{-2R_0}(1 + SNR_2) - 1)$ , the constraint on  $R_1 + R_2$  in (5) becomes redundant and the rate region reduces to (7). This concludes the achievability part of the proof.

Next, we prove a converse in the strong and very strong interference regimes. The converse is based on a technique used in [1] and [7] for proving the converse for the strong interference channel without the relay link. The main idea is to show that when  $INR_2 \ge SNR_2$ , if a rate pair  $(R_1, R_2)$  is achievable for the Gaussian Z-interference channel with a relay link, i.e.,  $X_1$  can be reliably decoded at receiver 1 at rate  $R_1$ , and  $X_2$  can be reliably decoded at receiver 2 at rate  $R_2$ , then  $X_2$  must also be decodable at the receiver 1.

First, the reliable decoding of  $X_2$  at  $Y_2$  requires

$$R_2 \le \gamma(\mathsf{SNR}_2). \tag{17}$$

To show that  $X_2$  is also decodable at  $Y_1$ , we note that after  $X_1$  is decoded at  $Y_1$  (possibly with the help of the relay

link), receiver 1 may now subtract  $X_1$  from  $Y_1$  then scale the resulting signal to obtain

$$Y_1' = \frac{h_{22}}{h_{21}}(Y_1 - h_{11}X_1) = h_{22}X_2 + \frac{h_{22}}{h_{21}}Z_1.$$
 (18)

When  $INR_2 \ge SNR_2$ , i.e.,  $h_{21} \ge h_{22}$ , the Gaussian noise  $\frac{h_{22}}{h_{21}}Z_1$  in this effective channel has a smaller variance than the noise in  $Y_2 = h_{22}X_2 + Z_2$ . Since  $X_2$  is reliably decodable at  $Y_2$ ,  $X_2$  must also be reliably decodable at  $Y_1$ .

Now, because both  $X_1$  and  $X_2$  are always decodable at  $Y_1$ in the strong interference regime, the achievable rate region of the Gaussian Z-interference channel with a digital relay link is equivalent to the capacity region of the same channel in which both  $X_1$  and  $X_2$  are *required* at  $Y_1$ , and  $X_2$  is required at  $Y_2$ . Further, the capacity region of such a channel can only be enlarged if  $X_2$  is given to  $Y_2$  by a genie. In such a case, the channel reduces to a Gaussian multiple-access channel with  $(X_1, X_2)$  as inputs,  $Y_1$  as the output, and with the same relay link from  $Y_2$  to  $Y_1$ , where the relay knows  $X_2$  perfectly. The capacity of such a multiple-access channel with a relay link is given by Lemma 1 in Appendix A:

$$\left\{ \left. \begin{pmatrix} R_1 &\leq \gamma(\mathsf{SNR}_1) \\ R_2 &\leq \gamma(\mathsf{INR}_2) + R_0 \\ R_1 + R_2 &\leq \gamma(\mathsf{SNR}_1 + \mathsf{INR}_2) + R_0 \end{cases} \right\}_{(19)}$$

Combining (19) and (17), then applying (16) gives us (5). Thus, we proved that when  $INR_2 \ge SNR_2$ , the achievable rate region of the Gaussian Z-interference channel with a relay link must be included in (5), which, in the very strong interference regime, reduces to (7).

#### C. Comments on Theorem 1

The achievability results for the classic Gaussian interference channel are categorized to the weak interference, the strong interference, and the very strong interference regimes. The capacity region of the classic interference channel in the weak interference case is still open. In the strong interference case, the capacity region is known as a pentagon. In the very strong interference case, the capacity region becomes a rectangle. The result of the previous section shows that the capacity region of the Type I Gaussian Z-relay-interference channel follows the same pattern. It is interesting to note that the relay link does not change the boundary between the weak and the strong interference regimes, but it does change the boundary between the strong and the very strong interference regimes. In other words, receiver-side relaying may potentially turn a strong interference channel into a very strong interference channel, but it never turns a weak interference channel into a strong interference channel.

It is instructive to compare the achievable regions of the Gaussian Z-interference channel with and without the relay link. Observe that when  $R_0 = 0$ , the achievable rate region (2) and the capacity region results (5) (7) reduce to previous results obtained in [1] and [7].

In the strong and very strong interference regimes, the capacity region is achieved by transmitting common information only at  $X_2$ . In the very strong interference regime, the relay

С

4.5

в

Fig. 7. Capacity region of the Gaussian Z-interference channel in the strong interference regime with and without a digital relay link of Type I.

link does not increase capacity, because the interference is already completely decoded and subtracted, even without the help of the relay.

In the strong interference regime, the relay link increases the capacity by helping the common information decoding at  $Y_1$ . In fact, a relay link of rate  $R_0$  increases the sum capacity by exactly  $R_0$  bits. Further, the increase in sum capacity can be arbitrarily divided between the two users, as long as the individual rates are below their respectively interference-free upper bound. This is because the relay link can either increase the common information rate (which improve the total rate at the interference-free receiver), or increase the power of the common information component of  $X_2$  (which decreases the noise at the interfered receiver), or do a combination of both.

As a numerical example, Fig. 7 shows the capacity region of a Gaussian Z-interference channel in the strong interference regime with and without the relay link. The channel parameters are set to be  $SNR_1 = SNR_2 = 25 dB$ ,  $INR_2 = 30 dB$ . The capacity region without the relay is the dash-dotted pentagon. With  $R_0 = 2$  bits, the capacity region expands to the dashed pentagon region, which represents an increase in sum rate of exactly 2 bits. As  $R_0$  increases to 4 bits, the channel falls into the very strong interference regime. The capacity region becomes the solid rectangular region.

The situation in the weak interference regime is more interesting. When  $INR_2 \leq SNR_2$ , the achievable rate region (2) is obtained by a Han-Kobayashi common-private power splitting scheme. By inspection, the effect of a relay link is to shift the rate region curve upward by  $R_0$  bits while limiting  $R_2$  by its single-user bound  $\gamma(SNR_2)$ . Interestingly, although the relay link of rate  $R_0$  is provided from the receiver 2 to the receiver 1, it can help  $R_2$  by exactly  $R_0$  bits, while it can only help  $R_1$  by strictly less than  $R_0$  bits!

The reason for this phenomenon is as follows. For the relay to help increase  $R_2$  while keeping  $R_1$  fixed,  $X_2$  must increase the common information rate. Since the common information is known at  $Y_2$ , relaying from  $Y_2$  to  $Y_1$  by binning achieves the cut-set bound-every bit in the relay link is worth one

Fig. 8. Achievable rate region of the Gaussian Z-interference channel in the weak interference regime with and without a digital relay link of Type I.

2

R, (bits)

2.5

3

3.5

4

bit in  $R_2$ . On the other hand, for the relay to help increase  $R_1$  while keeping  $R_2$  fixed,  $X_2$  needs to convert some of the private information into common information. In other words, the relay needs to use  $R_0$  to describe a larger portion of  $X_2$ , (which reduces the interference for user 1). The benefit of interference reduction to  $R_1$  is in general strictly less than  $R_0$ . But, as the next section shows, the benefit is asymptotically equal to  $R_0$  in the high SNR and high INR limit.

As a numerical example, Fig. 8 shows the achievable rate region of a Gaussian Z-interference channel in the weak interference regime with  $SNR_1 = SNR_2 = 25$ dB and  $INR_2 =$ 20dB. The solid curve represents the rate region achieved without the relay link. The dashed rate region is with a relay rate of  $R_0 = 2$  bits. For most part of the curve,  $R_0$  can be used to provide a 2-bit increase in  $R_2$ , but a less than 2-bit increase in  $R_1$ .

It is illustrative to identify the correspondence between the various points in the capacity region and different commonprivate splittings. Point A corresponds to  $\beta = 1$ . This is where the entire  $X_2$  is private message. In this case, it is easy to verify that the first term of  $R_2$  in (2) is less than the second term:

$$\gamma(\mathsf{SNR}_2) < \gamma(\beta \mathsf{SNR}_2) + \gamma\left(\frac{\overline{\beta}\mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta\mathsf{INR}_2}\right) + R_0 \tag{20}$$

As  $\beta$  decreases, more private message is converted into common message, which means that less interference is seen at receiver 1. As a result,  $R_1$  increases,  $R_2$  is kept at a constant (since (20) continues to hold). Graphically, as  $\beta$  decreases from 1, the achievable rate pair moves horizontally from point A to the right until it reaches point B, corresponding to some  $\beta^*$ , which is a critical point after which the second term of  $R_2$ in (2) becomes less than the first term  $\gamma(SNR_2)$ . The value of  $\beta^*$  can be obtained by solving the following equation:

$$\gamma(\mathsf{SNR}_2) = \gamma(\beta^*\mathsf{SNR}_2) + \gamma\left(\frac{\overline{\beta^*}\mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta^*\mathsf{INR}_2}\right) + R_0,$$



4.5

3.5

1.5

0.5

0 0

(bits) 2.5

щ

which yields

$$\beta^* = \frac{(1 + \mathsf{SNR}_1)(1 + \mathsf{SNR}_2) - 2^{2R_0}(1 + \mathsf{SNR}_1 + \mathsf{INR}_2)}{2^{2R_0}\mathsf{SNR}_2(1 + \mathsf{SNR}_1 + \mathsf{INR}_2) - \mathsf{INR}_2(1 + \mathsf{SNR}_2)}.$$
(21)

As  $\beta$  decreases further from  $\beta^*$ , more private message is converted into common message, which makes  $R_1$  even larger. However, when  $\beta < \beta^*$ , the amount of common message can be transmitted is restricted by the interference link  $h_{21}$ and the relay link rather than the direct link  $h_{22}$  of user 2. Therefore, user 2's data rate cannot be kept as a constant;  $R_2$ also goes down as user 1's rate goes up. As shown in Fig. 8, the achievable rate pair moves from point B to point C as  $\beta$  decreases from  $\beta^*$  to 0. Point C corresponds to where the entire  $X_2$  is common message.

#### D. Asymptotic Sum Capacity

Practical communication systems often operate in the interference-limited regime, where both the signal and the interference are much stronger than the receiver thermal noise. In this section, we investigate the asymptotic sum capacity of the Type I Gaussian Z-relay-interference channel in the weak interference regime when

$$\min\{\mathsf{SNR}_1, \mathsf{SNR}_2, \mathsf{INR}_2\} \gg 1.$$
(22)

More precisely, we let noise power  $N \rightarrow 0$ , while keeping power constraints  $P_1$ ,  $P_2$ , channel parameters  $h_{ij}$ , and the digital relay link rate  $R_0$  fixed. In other words,  $SNR_1$ ,  $SNR_2$ ,  $INR_2 \rightarrow \infty$ , while their ratios are kept constant.

Denote the sum capacity of a Type I Gaussian Z-interference channel with a relay link of rate  $R_0$  by  $C_{sum}(R_0)$ . Without the digital relay link, or equivalently  $R_0 = 0$ , the sum capacity of the classic Gaussian Z-interference channel in the weak interference regime (i.e.  $INR_2 \leq SNR_2$ ) is given by [8], [5]:

$$C_{sum}(0) = \gamma(\mathsf{SNR}_2) + \gamma\left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right)$$
(23)

which is achieved by independent Gaussian codebooks and treating the interference as noise at the receiver. In the high SNR/INR limit, the above sum capacity becomes

$$C_{sum}(0) \approx \frac{1}{2} \log \left( \frac{\mathsf{SNR}_2(\mathsf{SNR}_1 + \mathsf{INR}_2)}{\mathsf{INR}_2} \right)$$
 (24)

where the notation  $f(x) \approx g(x)$  is used to denote  $\lim f(x) - g(x)$ q(x) = 0. In the above expression, the limit is taken as  $N \to 0$ .

With a digital relay link of finite capacity  $R_0$ , how many bits can it contribute to the sum rate? Intuitively, the sum rate increase due to the relay link must be bounded by  $R_0$ . In the following, we show that in the high SNR/INR limit, the asymptotic sum capacity increase is in fact  $R_0$  in the weakinterference regime.

Theorem 2: For the Type I Gaussian Z-interference channel with a digital relay link of limited rate  $R_0$  from the interference-free receiver to the interfered receiver as shown in Fig. 1(a), when  $INR_2 \leq SNR_2$ , the asymptotic sum capacity is given by

$$C_{sum}(R_0) \approx C_{sum}(0) + R_0. \tag{25}$$

*Proof:* We first prove the achievability. As illustrated in Fig. 5 the sum rate of the Type I Gaussian Z-relay-interference channel is achieved with  $\beta = \beta^*$ , where  $\beta^*$  is as derived in (21). In the high SNR/INR limit, we have

$$\lim_{N \to 0} \beta^* = \frac{2^{-2R_0}}{1 + (1 - 2^{-2R_0}) \frac{\mathsf{INR}_2}{\mathsf{SNR}_1}}.$$
 (26)

Substituting this  $\beta^*$  into the achievable rate pair in (2), we obtain the asymptotic rate pair as

$$\begin{cases} R_1 \approx \frac{1}{2} \log \left( 1 + \frac{\mathsf{SNR}_1}{\mathsf{INR}_2} \right) + R_0 \\ R_2 \approx \frac{1}{2} \log(\mathsf{SNR}_2) \end{cases}$$
(27)

which gives the following asymptotic sum rate:

$$R_{sum} \approx \frac{1}{2} \log \left( \frac{\mathsf{SNR}_2(\mathsf{SNR}_1 + \mathsf{INR}_2)}{\mathsf{INR}_2} \right) + R_0$$
$$\approx C_{sum}(0) + R_0. \tag{28}$$

The converse proof starts with Fano's inequality. Denote the output of the digital relay link over the *n*-block by  $V^n$ . Since the digital link has a capacity limit  $R_0$ ,  $V^n$  is a discrete random variable with  $H(V^n) \leq nR_0$ . For any codebook of block length n, we have

$$n(R_{1} + R_{2})$$

$$\stackrel{(a)}{\leq} I(X_{1}^{n}; Y_{1}^{n}, V^{n}) + I(X_{2}^{n}; Y_{2}^{n}) + n\epsilon_{n}$$

$$= I(X_{1}^{n}; Y_{1}^{n}) + I(X_{1}^{n}; V^{n}|Y_{1}^{n}) + I(X_{2}^{n}; Y_{2}^{n}) + n\epsilon_{n}$$

$$\stackrel{(b)}{\leq} I(X_{1}^{n}; Y_{1}^{n}) + H(V^{n}|Y_{1}^{n}) + I(X_{2}^{n}; Y_{2}^{n}) + n\epsilon_{n}$$

$$\stackrel{(c)}{\leq} I(X_{1}^{n}; Y_{1}^{n}) + I(X_{2}^{n}; Y_{2}^{n}) + nR_{0} + n\epsilon_{n}$$

$$= h(Y_{1}^{n}) - h(Y_{1}^{n}|X_{1}^{n}) + h(Y_{2}^{n}) - h(Z_{2}^{n}) + nR_{0} + n\epsilon_{n}$$

$$= h(Y_{1}^{n}) - h(Z_{2}^{n}) + h(h_{22}X_{2}^{n} + Z_{2}^{n}) - h(h_{21}X_{2}^{n} + Z_{1}^{n}) + nR_{0} + n\epsilon_{n}$$

$$\stackrel{(d)}{=} h(Y_{1}^{n}) - h(Z_{2}^{n}) + h(X_{2}^{n} + \tilde{Z}_{2}^{n}) - h(X_{2}^{n} + \tilde{Z}_{1}^{n}) + n\log\frac{|h_{22}|^{2}}{|h_{21}|^{2}} + nR_{0} + n\epsilon_{n}$$

$$(29)$$

where

- (a) follows from Fano's inequality;
- is due to  $H(V^n | X_1^n, Y_1^n) \ge 0;$ (b)
- (c)
- is due to  $H(V^n|Y_1^n) \leq H(V^n) \leq nR_0$ ; is where  $\tilde{Z}_1^n$  and  $\tilde{Z}_2^n$  are defined as  $\tilde{Z}_1^n =$ (d)  $Z_1^n/h_{21}, \tilde{Z}_2^n = Z_2^n/h_{22}.$

Since  $Z_1$  and  $Z_2$  are i.i.d. Gaussian random vectors with the same variance, when  $h_{21} \leq h_{22}$  (or equivalently,  $INR_2 \leq SNR_2$ ,  $Z_1$  has a larger variance than  $Z_2$ . In this case, we can use an extremal inequality due to Liu and Viswanath [18] to show that under an average power constraint  $nP_2$  on  $X_2^n$ ,  $h(X_2^n + \tilde{Z}_2^n) - h(X_2^n + \tilde{Z}_1^n)$  is maximized when  $X_2^n$  is an i.i.d. Gaussian random vector with distribution  $\mathcal{N}(0, P_2)$ . This technique was used earlier by Motahari and Khandani [3, Lemma 1].

Now, consider  $Y_1^n = h_{11}X_1^n + h_{21}X_2^n + Z_1^n$ . Note that  $h(Y_1^n)$  is maximized under power constraints  $nP_1$  on  $X_1^n$  and  $nP_2$  on  $X_2^n$  when  $X_1^n$  and  $X_2^n$  are i.i.d. Gaussian distributed as  $\mathcal{N}(0, P_1)$  and  $\mathcal{N}(0, P_2)$ , respectively. Since the same i.i.d. distribution  $\mathcal{N}(0, P_1)$  for  $X_1^n$  maximizes both  $h(Y_1^n)$  and  $h(X_2^n + \tilde{Z}_2^n) - h(X_2^n + \tilde{Z}_1^n)$  simultaneously, it must maximize the entire right-hand side of (29).

Evaluating right-hand side of (29) with  $X_1 \sim \mathcal{N}(0, P_1)$  and  $X_2 \sim \mathcal{N}(0, P_2)$ , we obtain the following upper bound for the sum rate:

$$R_1 + R_2 \leq \gamma(\mathsf{SNR}_2) + \gamma\left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + R_0 + \epsilon_n$$
  
=  $C_{sum}(0) + R_0 + \epsilon_n$  (30)

where  $\epsilon_n \to 0$  as n goes to infinity. Note that this upper bound holds for all ranges of SNR<sub>1</sub>, SNR<sub>2</sub>, and INR<sub>2</sub> as long as INR<sub>2</sub>  $\leq$  SNR<sub>2</sub>. This, when combined with the asymptotic achievability result proved earlier, gives the asymptotic sum capacity  $C_{sum}(R_0) \approx C_{sum}(0) + R_0$ .

The above result shows that the partial-interferenceforwarding strategy essentially meets the sum-rate cut-set bound asymptotically at high SNR/INR. Each bit in the digital relay link is worth one bit to the sum capacity.

We have thus far focused on the sum-capacity achieving power splitting ratio  $\beta^*$ . As  $\beta \leq \beta^*$ , the achievable rate pair goes from point *B* to point *C* along the dashed curve as shown in Fig. 8. It turns out that for any fixed  $\beta \leq \beta^*$ , the rate sum asymptotically approaches the sum capacity upper bound. To see this, fix some arbitrary  $\beta \leq \beta^*$ , the sum rate corresponding to this  $\beta$  is given in Theorem 1 by:

$$R_{sum} = \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right) + \gamma(\beta \mathsf{SNR}_2) + \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + R_0 = \frac{1}{2} \log \left( \frac{1 + \beta \mathsf{SNR}_2}{1 + \beta \mathsf{INR}_2} \right) + \gamma(\mathsf{SNR}_1 + \mathsf{SNR}_2) + R_0.$$
(31)

Let  $N \rightarrow 0$ , then eventually

$$\beta \min\{\mathsf{SNR}_2, \mathsf{INR}_2\} \gg 1. \tag{32}$$

In this case, the sum rate goes to

$$R_{sum} \approx \frac{1}{2} \log \left( \frac{\mathsf{SNR}_2(\mathsf{SNR}_1 + \mathsf{INR}_2)}{\mathsf{INR}_2} \right) + R_0, \qquad (33)$$

which is again the asymptotic sum capacity. This calculation implies that in the high SNR/INR regime, the dashed curve in Fig. 8 has an initial slope of -1 as  $\beta$  goes from  $\beta^*$  to 0.

Interestingly, partial interference forwarding is not the only way to asymptotically achieve the sum capacity of the Type I Gaussian Z-relay-interference channel. In the following, we show that a quantize-and-forward relaying scheme is also asymptotically sum capacity achieving.

In the quantize-and-forward scheme, no common-private information splitting is performed. Each receiver only decodes the message intended for it. As shown in Fig. 9, receiver 2 compresses its received signal  $Y_2$  into  $\hat{Y}_2$ , then forwards it to the receiver 1 through the digital link  $R_0$ .



Fig. 9. Quantize-and-forward in Gaussian Z-interference relay channel.

Clearly, the rate of user 2 is given by

$$R_2 = \max_{p(x_2)} I(X_2; Y_2). \tag{34}$$

Using the Wyner-Ziv coding strategy for the relay channel [19], [9], for a fixed  $p(x_2)$ , the following rate for user 1 is achievable:

$$R_1 = \max_{p(x_1)p(\hat{y}_2|y_2)} I(X_1; Y_1, \hat{Y}_2)$$
(35)

under the constraint

$$I(Y_2; \hat{Y}_2 | Y_1) \le R_0. \tag{36}$$

The optimization in (35) is in general hard. Here, we adopt independent Gaussian codebooks with  $X_1 \sim \mathcal{N}(0, P_1)$  and  $X_2 \sim \mathcal{N}(0, P_2)$ , and a Gaussian quantization scheme for the compression of  $Y_2$ :

$$\ddot{Y}_2 = Y_2 + e \tag{37}$$

where e is a Gaussian random variable independent of  $Y_2$ , with a distribution  $\mathcal{N}(0, \sigma^2)$ . We show in Appendix C that this choice of  $p(x_1)p(x_2)p(\hat{y}_2|y_2)$  gives the following achievable rate pair:

$$\begin{cases} R_1 = \gamma \left( \frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2} \right) + R_0 - \delta_0(R_0) \\ R_2 = \gamma(\mathsf{SNR}_2) \end{cases}$$
(38)

where

$$\begin{split} \delta_0(R_0) &= \\ \gamma \left( \frac{(2^{2R_0} - 1)(1 + \mathsf{SNR}_2 + \mathsf{INR}_2)(1 + \mathsf{SNR}_1 + \mathsf{INR}_2)}{(1 + \mathsf{INR}_2)((1 + \mathsf{SNR}_1)(1 + \mathsf{SNR}_2) + \mathsf{INR}_2)} \right) \end{split}$$

Let  $N \rightarrow 0$ , the above rate pair asymptotically goes to

$$\begin{cases} R_1 \approx \frac{1}{2} \log \left( 1 + \frac{\mathsf{SNR}_1}{\mathsf{INR}_2} \right) + R_0 \\ R_2 \approx \frac{1}{2} \log(\mathsf{SNR}_2) \end{cases}$$
(39)

which is the same as the asymptotic rate pair achieved with  $\beta = \beta^*$  using the partial-interference-forwarding scheme. Therefore, quantize-and-forward also achieves the asymptotic sum capacity (25). We remark that this is akin to the capacity result for a class deterministic relay channel [10], where both bin-and-forward and quantize-and-forward are shown to be capacity achieving.

# III. GAUSSIAN Z-INTERFERENCE CHANNEL WITH A RELAY LINK: TYPE II

As a counterpart of the Type I channel considered in the previous section, this section studies the Type II Gaussian Z-relay-interference channel, where the relay link goes from the interfered receiver to the interference-free receiver as shown in Fig. 1(b). Intuitively, when the interference link is weak, the digital link would not be as efficient as in the Type I channel, because receiver 1's knowledge of  $X_2$  is inferior to that of the receiver 2. However, when the interference link is very strong, receiver 1 becomes a better receiver for  $X_2$  than receiver 2, in which case the digital link is capable of increasing user 2's rate by as much as  $R_0$ . This section makes these intuitions precise by deriving an achievability theorem for the weak and moderately strong interference regimes, and a capacity theorem for the strong and very strong interference regimes.

A significant difference between the Type I and Type II channel models is that, in the Type I channel, the relay  $(Y_2)$  observes a noisy version of the *interference* at the relay destination  $(Y_1)$ , thus partial interference forwarding, which exploits the codebook structure of the interference signal is a natural relay strategy. In the Type II channel, the relay  $(Y_1)$ now observes a noisy version of the intended signal at the relay destination  $(Y_2)$ . Thus, decode-and-forward and quantize-andforward become natural relay strategies. The main result of this section is an achievability region based on a combination of these two relay strategies and the Han-Kobayashi commonprivate information splitting scheme, whereas the relay decodes the common information and forwards the bin index using a part of the digital link and forwards quantization information of the private information using the other part of the digital link.

#### A. Achievable Rate Region

Theorem 3: For the Type II Gaussian Z-interference channel with a digital relay link of limited rate  $R_0$  from the interfered receiver to the interference-free receiver as shown in Fig. 1(b), in the weak interference regime defined by  $INR_2 \leq SNR_2$ , the following rate region is achievable

$$\bigcup_{0 \le \beta \le 1} \left\{ (R_1, R_2) \left| R_1 \le \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right), \right. \\ \left. R_2 \le \gamma(\beta \mathsf{SNR}_2) + \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + \delta(\beta, R_0) \right\},$$
(40)

where

$$\delta(\beta, R_0) = \gamma \left( \frac{\beta(2^{2R_0} - 1)\mathsf{INR}_2}{2^{2R_0}(1 + \beta\mathsf{SNR}_2) + \beta\mathsf{INR}_2} \right).$$
(41)

In the moderately strong interference regime, defined by

$$\mathsf{SNR}_2 \le \mathsf{INR}_2 \le 2^{2R_0}(1+\mathsf{SNR}_2) - 1 \stackrel{\triangle}{=} \mathsf{INR}_2^{\dagger},$$
 (42)

the following rate region is achievable:

$$\cos\left\{\bigcup_{\alpha\in\mathbb{R},0\leq\beta\leq1,\ R_a+R_b\leq R_0}\mathcal{R}_{\alpha,\beta}(R_a,R_b)\right\}$$
(43)

١



Fig. 10. Common-private information splitting for Type II channel.

where "co" denotes convex hull and  $\mathcal{R}_{\alpha,\beta}(R_a, R_b)$  is a pentagon region given by

$$\left\{ (R_1, R_2) \middle| \begin{array}{c} R_1 \leq \gamma(\frac{\mathsf{SNR}_1}{1+\beta\mathsf{INR}_2}) \\ R_2 \leq \min\left\{\gamma(\mathsf{SNR}_2) + R_b + \eta(\alpha, \beta, R_a), \\ \gamma(\beta\mathsf{SNR}_2) + \gamma\left(\frac{\overline{\beta}\mathsf{INR}_2}{1+\beta\mathsf{INR}_2}\right) \\ + \zeta(\alpha, \beta, R_a)\right\} \\ R_1 + R_2 \leq \gamma(\beta\mathsf{SNR}_2) + \gamma\left(\frac{\mathsf{SNR}_1 + \overline{\beta}\mathsf{INR}_2}{1+\beta\mathsf{INR}_2}\right) \\ + \zeta(\alpha, \beta, R_a) \end{array} \right\} \\ \left. \begin{array}{c} (44) \end{array} \right\}$$

where

$$\zeta(\alpha,\beta,R_a) = \gamma\left(\frac{\beta \mathsf{INR}_2}{(1+\beta \mathsf{SNR}_2)(1+\frac{\sigma^2}{N})}\right), \quad (45)$$

and

$$\eta(\alpha, \beta, R_a) = \gamma\left(\frac{(1+2\alpha\overline{\beta}+\alpha^2\overline{\beta})\mathsf{INR}_2+\beta\overline{\beta}\alpha^2\mathsf{INR}_2\mathsf{SNR}_2}{(1+\mathsf{SNR}_2)(1+\frac{\sigma^2}{N})}\right) \quad (46)$$

with

$$\frac{\sigma^2}{N} = \frac{1 + \mathsf{SNR}_2 + (1 + 2\alpha\overline{\beta} + \alpha^2\overline{\beta})\mathsf{INR}_2 + \beta\overline{\beta}\alpha^2\mathsf{INR}_2\mathsf{SNR}_2}{(2^{2R_a} - 1)(1 + \mathsf{SNR}_2)}$$
(47)

In the strong interference regime defined by

$$\mathsf{INR}_2^{\dagger} \le \mathsf{INR}_2 \le (1 + \mathsf{SNR}_1)\mathsf{INR}_2^{\dagger} \stackrel{\triangle}{=} \mathsf{INR}_2^{\ddagger}, \qquad (48)$$

the capacity region is given by

$$\left\{ (R_1, R_2) \middle| \begin{array}{rcl}
R_1 &\leq & \gamma(\mathsf{SNR}_1) \\
R_2 &\leq & \gamma(\mathsf{SNR}_2) + R_0 \\
R_1 + R_2 &\leq & \gamma(\mathsf{SNR}_1 + \mathsf{INR}_2) \end{array} \right\}. \quad (49)$$

In the very strong interference regime defined by

$$\mathsf{INR}_2 \ge \mathsf{INR}_2^{\ddagger},\tag{50}$$

the capacity region is given by

$$\left\{ \begin{pmatrix} R_1, R_2 \end{pmatrix} \middle| \begin{array}{c} R_1 \le \gamma(\mathsf{SNR}_1) \\ R_2 \le \gamma(\mathsf{SNR}_2) + R_0 \\ \end{array} \right\}.$$
(51)

*Proof:* We first prove the achievability of the rate region given in (43). it will be shown that this is an achievable rate region not only for the moderately strong interference regime, but also for any INR and SNR. We then show that (43) reduces to (40) in the weak interference regime, and reduces to (49) and (51) in the strong and very strong interference regimes, respectively.

The achievability of (43) is based on the Han-Kobayashi common-private information splitting scheme and the same

 $Z \sim \mathcal{N}(0, N)$ 

 $Y_2$ 

Fig. 11. Gaussian multiple-access channel with two digital relay links.

 $U_2$ 

codebooks as in Theorem 1. The encoding procedure is as shown in Fig. 10.

A two-step decoding procedure is used. Consider first the decoding of  $(X_1, W_2)$  at  $Y_1$ . The achievable set of  $(S_1, T_2)$  is the capacity region of a multiple-access channel, denoted by  $C_1$ , where

$$\begin{cases} S_{1} \leq \gamma \left( \frac{\mathsf{SNR}_{1}}{1 + \beta \mathsf{INR}_{2}} \right) \\ T_{2} \leq \gamma \left( \frac{\overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}} \right) \\ S_{1} + T_{2} \leq \gamma \left( \frac{\mathsf{SNR}_{1} + \overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}} \right). \end{cases}$$
(52)

Next, consider the decoding of  $(W_2, U_2)$  at  $Y_2$  with the help of a digital relay link of rate  $R_0$ . This is a multipleaccess channel with a rate-limited relay, where the relay has complete knowledge of  $W_2$  and a noisy observation  $h_{21}U_2 + Z_1$ , obtained by subtracting  $X_1$  and  $W_2$  from the received signal at  $Y_1$ . Each of these two pieces of information is useful for decoding  $(W_2, U_2)$  at  $Y_2$ .

We now consider a relay scheme which splits the digital link in two parts:  $R_a$  bits of the digital link for describing  $W_2$ , and  $R_b$  for describing  $U_2$ , where  $R_a + R_b = R_0$ . Since  $W_2$  is perfectly known at the relay, a straightforward binning technique of decode-and-forward can be used to describe  $W_2$ . By Lemma 1 of Appendix A, a relay link of rate  $R_b$ enlarges the capacity region of the multiple-access channel by increasing the rate for  $W_2$  and the sum rate by exactly  $R_b$ .

On the other hand, since only a noisy version of  $U_2$  is available at the relay, a quantize-and-forward strategy using Wyner-Ziv coding ([19], [9]) may be used for describing  $U_2$ . A straightforward application of quantize-and-forward would be to quantize  $h_{21}U_2 + Z_1$  with  $Y_2$  used as the decoder side information. However, the presence of  $W_2$  offers other possibilities. First, the decoder of the multiple-access channel may choose to decode  $W_2$  before decoding  $U_2$ , in which case  $W_2$  becomes an additional decoder side information for Wyner-Ziv coding. Second, instead of quantizing  $h_{21}U_2 + Z_1$ with  $W_2$  completely subtracted from the relay's observation, the relay may choose to subtract  $W_2$  partially—doing so can benefit the Wyner-Ziv rate. This second approach is adopted for the rest of the proof. Interestingly, as will be verified later, the second approach turns out to include the first approach as a special case.

More specifically, let the relay form the following fictitious signal

$$\bar{Y}_1 = h_{21}(U_2 + W_2) + \alpha h_{21}W_2 + Z_1$$
(53)

for some  $\alpha \in \mathbb{R}$ . The proposed relay scheme, which combines the decode-and-forward technique and the quantize-andforward technique, is now illustrated in Fig. 11, where  $W_2$  and  $U_2$  are the inputs of the multiple-access channel,  $(Y_2, \hat{Y}_1)$  is the output, and  $\hat{Y}_1$  is a quantized version of  $\bar{Y}_1$ . With complete knowledge of  $W_2$  at the relay, by Lemma 1 in Appendix A, the capacity of this multiple-access relay channel, denoted by  $C_2$ , is given by the set of rates  $(S_2, T_2)$  where

$$\begin{cases} S_2 \leq I(U_2; Y_2, \hat{Y}_1 | W_2) \\ T_2 \leq I(W_2; Y_2, \hat{Y}_1 | U_2) + R_b \\ S_2 + T_2 \leq I(U_2, W_2; Y_2, \hat{Y}_1) + R_b \end{cases}$$
(54)

Similar to Theorem 1, we adopt a Gaussian quantization scheme to quantize  $\bar{Y}_1$ :

$$\hat{Y}_1 = \bar{Y}_1 + e \tag{55}$$

where e is a Gaussian random variable independent of  $\bar{Y}_1$ , with a distribution  $\mathcal{N}(0, \sigma^2)$ . With the encoder side information  $W_2$ at the input of the relay link and the decoder side information  $Y_2$  at the output of the relay link, the Wyner-Ziv coding rate for quantizing  $\bar{Y}_1$  into  $\hat{Y}_1$  is given by [20] [9]

$$I(\hat{Y}_1; W_2, \bar{Y}_1) - I(\hat{Y}_1; Y_2) \le R_a.$$
(56)

But

$$I(Y_{1}; W_{2}, Y_{1}) - I(Y_{1}; Y_{2})$$

$$= I(\hat{Y}_{1}; \bar{Y}_{1}) + I(\hat{Y}_{1}; W_{2} | \bar{Y}_{1}) - I(\hat{Y}_{1}; Y_{2})$$

$$\stackrel{(a)}{=} I(\hat{Y}_{1}; \bar{Y}_{1}) - I(\hat{Y}_{1}; Y_{2})$$

$$\stackrel{(b)}{=} I(\hat{Y}_{1}; \bar{Y}_{1} | Y_{2})$$
(57)

where both (a) and (b) come from the fact that  $\hat{Y}_1 = \bar{Y}_1 + e$ and e is independent of  $W_2$  or  $Y_2$ . Thus, the rate constraint becomes

$$I(Y_1; Y_1 | Y_2) \le R_a.$$
 (58)

To fully utilize the channel, we set  $\hat{Y}_1$  to be such that  $I(\hat{Y}_1; \bar{Y}_1|Y_2)$  is equal to  $R_a$ . To find  $\sigma^2$ , note that

$$R_{a} = I(\hat{Y}_{1}; \bar{Y}_{1}|Y_{2})$$

$$= h(\hat{Y}_{1}|Y_{2}) - h(\hat{Y}_{1}|\bar{Y}_{1}, Y_{2})$$

$$= \frac{1}{2}\log(2\pi e\sigma_{\hat{Y}_{1}|Y_{2}}^{2}) - \frac{1}{2}\log(2\pi e\sigma^{2})$$

$$= \frac{1}{2}\log\left(\frac{\sigma_{\hat{Y}_{1}|Y_{2}}^{2}}{\sigma^{2}}\right)$$
(59)

where  $\sigma^2_{\hat{Y}_1|Y_2}$ , the conditional minimum mean-squared error (MMSE) of  $\hat{Y}_1$  given  $Y_2$ , can be calculated as

$$\begin{split} \sigma_{\hat{Y}_{1}|Y_{2}}^{2} &= \sigma_{\hat{Y}_{1}}^{2} - \sigma_{\hat{Y}_{1},Y_{2}}(\sigma_{Y_{2}}^{2})^{-1}\sigma_{Y_{2},\hat{Y}_{1}} \\ &= (\beta\overline{\beta}\alpha^{2}|h_{21}|^{2}|h_{22}|^{2}P_{2}^{2} + (|h_{22}|^{2}P_{2} + N)(N + \sigma^{2}) + \\ &\quad (1 + 2\alpha\overline{\beta} + \alpha^{2}\overline{\beta})P_{2}|h_{21}|^{2}N)/(|h_{22}|^{2}P_{2} + N). \end{split}$$

Substituting the above  $\sigma_{\hat{Y}_1|Y_2}^2$  into (59), we obtain (47). Now, we evaluate the multiple-access relay channel capacity

Now, we evaluate the multiple-access relay channel capacity region  $C_2$  in (54). Define

$$\begin{cases} I(U_2; \hat{Y}_1 | Y_2, W_2) \stackrel{\triangle}{=} \zeta(\alpha, \beta, R_a) \\ I(W_2; \hat{Y}_1 | Y_2, U_2) \stackrel{\triangle}{=} \xi(\alpha, \beta, R_a) \\ I(W_2, U_2; \hat{Y}_1 | Y_2) \stackrel{\triangle}{=} \eta(\alpha, \beta, R_a). \end{cases}$$
(61)

Applying Gaussian distributions  $W_2 \sim \mathcal{N}(0, \overline{\beta}P_2)$  and  $U_2 \sim \mathcal{N}(0, \beta P_2)$ ,  $C_2$  becomes

$$\begin{cases} S_2 \leq \gamma(\beta \mathsf{SNR}_2) + \zeta(\alpha, \beta, R_a) \\ T_2 \leq \gamma(\overline{\beta} \mathsf{SNR}_2) + \xi(\alpha, \beta, R_a) + R_b \\ S_2 + T_2 \leq \gamma(\mathsf{SNR}_2) + \eta(\alpha, \beta, R_a) + R_b. \end{cases}$$
(62)

The computations of  $\zeta(\alpha, \beta, R_a)$ ,  $\xi(\alpha, \beta, R_a)$  and  $\eta(\alpha, \beta, R_a)$  again involve Gaussian MMSE expressions. First,

$$\eta(\alpha, \beta, R_a) = h(\hat{Y}_1 | Y_2) - h(\hat{Y}_1 | U_2, W_2, Y_2) = \frac{1}{2} \log \left( \frac{\sigma_{\hat{Y}_1 | Y_2}^2}{N + \sigma^2} \right).$$
(63)

Substituting (60) into the above, we obtain (46). Likewise,

$$\begin{aligned} \zeta(\alpha, \beta, R_a) &= h(\hat{Y}_1 | Y_2, W_2) - h(\hat{Y}_1 | U_2, W_2, Y_2) \\ &= \frac{1}{2} \log \left( \frac{\sigma_{\hat{Y}_1 | Y_2, W_2}^2}{N + \sigma^2} \right). \end{aligned}$$
(64)

A similar computation leads to (45). The computation of  $\xi(\alpha, \beta, R_a)$  can be done in a similar fashion, but the detailed expression does not affect our final result.

Finally, an achievable rate region for the Gaussian Zinterference channel with a relay link is a set of  $(R_1, R_2)$ with  $R_1 = S_1$  and  $R_2 = S_2 + T_2$ , for which  $(S_1, T_2) \in C_1$ and  $(S_2, T_2) \in C_2$ . Combining the  $C_1$  region (52) and the  $C_2$ region (62) in the same manner as in the proof of Theorem 1, we obtain a pentagon achievable rate region  $R_{\alpha,\beta}(R_a, R_b)$  for each fixed  $\alpha$ ,  $0 \leq \beta \leq 1$  and  $R_a + R_b = R_0$ :

$$\left\{ (R_1, R_2) \middle| \begin{array}{c} R_1 \leq \gamma(\frac{\mathsf{SNR}_1}{1+\beta\mathsf{INR}_2}) \\ R_2 \leq \min\left\{\gamma(\mathsf{SNR}_2) + R_b + \eta(\alpha, \beta, R_a), \\ \gamma(\beta\mathsf{SNR}_2) + \gamma\left(\frac{\overline{\beta\mathsf{INR}}_2}{1+\beta\mathsf{INR}_2}\right) \\ + \zeta(\alpha, \beta, R_a)\right\} \\ R_1 + R_2 \leq \gamma(\beta\mathsf{SNR}_2) + \gamma\left(\frac{\mathsf{SNR}_1 + \overline{\beta}\mathsf{INR}_2}{1+\beta\mathsf{INR}_2}\right) \\ + \zeta(\alpha, \beta, R_a) \end{array} \right\}.$$

Therefore, the overall achievable rate region is

$$\operatorname{co}\left\{\bigcup_{\alpha\in\mathbb{R},0\leq\beta\leq1,\ R_{a}+R_{b}\leq R_{0}}\mathcal{R}_{\alpha,\beta}(R_{a},R_{b})\right\}$$
(66)

This concludes the proof of the general achievability region (43).

In the following, we show that (40), (49) and (51) are all included in the above achievable rate region.

First, consider the weak interference regime, where  $INR_2 \leq SNR_2$ . For any nonnegative  $R_b$  and when  $INR_2 \leq SNR_2$ , it is easy to verify the following two inequalities:

$$\gamma(\beta \mathsf{SNR}_2) + \gamma\left(\frac{\overline{\beta}\mathsf{INR}_2}{1 + \beta\mathsf{INR}_2}\right) \le \gamma(\mathsf{SNR}_2) + R_b \qquad (67)$$

and

$$\zeta(\alpha, \beta, R_a) \le \eta(\alpha, \beta, R_a). \tag{68}$$

Thus, the second term in the minimization in the expression of  $R_2$  in (65) is always less than the first term. As a result,  $\mathcal{R}_{\alpha,\beta}(R_a, R_b)$  simplifies to

$$\left\{ \left. (R_1, R_2) \middle| \begin{array}{l} R_1 \leq \gamma(\frac{\mathsf{SNR}_1}{1+\beta\mathsf{INR}_2}) \\ R_2 \leq \gamma(\beta\mathsf{SNR}_2) + \gamma\left(\frac{\overline{\beta}\mathsf{INR}_2}{1+\beta\mathsf{INR}_2}\right) \\ +\zeta(\alpha, \beta, R_a) \\ R_1 + R_2 \leq \gamma(\beta\mathsf{SNR}_2) + \gamma\left(\frac{\mathsf{SNR}_1 + \overline{\beta}\mathsf{INR}_2}{1+\beta\mathsf{INR}_2}\right) \\ +\zeta(\alpha, \beta, R_a) \end{array} \right\}.$$
(69)

Inspecting the above region, we see that  $R_a$  enters the above expression only through  $\zeta(\alpha, \beta, R_a)$ . It is easy to verify that  $\zeta(\alpha, \beta, R_a)$  is a monotonically nondecreasing function of  $R_a$ . Thus, the maximum achievability region is obtained for  $R_a = R_0$  and  $R_b = 0$ . This means that a pure quantization scheme is optimal in the weak interference regime.

Further,  $\alpha$  enters the rate region expression also only through  $\zeta(\alpha, \beta, R_0)$ . Thus, we can choose  $\alpha$  to maximize  $\zeta(\alpha, \beta, R_0)$ , or equivalently, to minimize  $\sigma^2$  in (47). Taking the derivative of (47) with respect to  $\alpha$  and setting it to zero, the optimal  $\alpha$  is

$$\alpha^* = -\frac{1}{1 + \beta \mathsf{SNR}_2}.\tag{70}$$

Substituting  $\alpha^*$  into (47), we obtain

$$\frac{\sigma^2}{N} = \frac{1}{2^{2R_0} - 1} \left( 1 + \frac{\beta \mathsf{INR}_2}{1 + \beta \mathsf{SNR}_2} \right), \tag{71}$$

which gives a derivation of (41):

$$\zeta(\alpha^*, \beta, R_0) = \gamma \left(\frac{\beta(2^{2R_0} - 1)\mathsf{INR}_2}{2^{2R_0}(1 + \beta\mathsf{SNR}_2) + \beta\mathsf{INR}_2}\right) \stackrel{\triangle}{=} \delta(\beta, R_0).$$
(72)

Finally, we take the union of all  $\mathcal{R}_{\alpha^*,\beta}(R_0,0)$ . Substitute the above  $\delta(\beta, R_0)$  into (69) and denote the rate constraints of the pentagon as

$$f_{1}(\beta) = \gamma \left(\frac{\mathsf{SNR}_{1}}{1 + \beta \mathsf{INR}_{2}}\right)$$
  

$$f_{2}(\beta) = \gamma \left(\beta \mathsf{SNR}_{2}\right) + \gamma \left(\frac{\overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}}\right) + \delta(\beta, R_{0})$$
  

$$f_{3}(\beta) = \gamma \left(\beta \mathsf{SNR}_{2}\right) + \gamma \left(\frac{\mathsf{SNR}_{1} + \overline{\beta} \mathsf{INR}_{2}}{1 + \beta \mathsf{INR}_{2}}\right) + \delta(\beta, R_{0}).$$

It is easy to check that, first, the pentagon reduces to a rectangle when  $\beta = 1$ . Second,  $f_2(\beta) - f_3(\beta)$  is a constant. Third, when  $INR_2 \leq SNR_2$ , we have  $f'_1(\beta) < 0$  and  $f'_2(\beta) = f'_3(\beta) \geq 0$ . Therefore, as  $\beta$  decreases from 1 to 0,  $f_1(\beta)$  is monotonically increasing, while both  $f_2(\beta)$  and  $f_3(\beta)$  are monotonically decreasing. This is exactly the same situation as the proof of Theorem 1; see Fig. 5. Consequently, the union of achievable pentagons,  $\bigcup_{0 \leq \beta \leq 1} \mathcal{R}_{\alpha^*,\beta}(R_0,0)$  is defined by  $R_1 \leq \gamma(SNR_1), R_2 \leq \gamma(SNR_2) + \delta(\beta, R_0)$ , and lower-right corner points of the pentagons

$$\begin{cases} R_1 = \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right) \\ R_2 = \gamma (\beta \mathsf{SNR}_2) + \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + \delta(\beta, R_0). \end{cases}$$
(73)

We prove in Appendix D that this region is convex when  $INR_2 \leq SNR_2$ . Therefore, the convex hull is not needed. This establishes the rate region (40) for the weak interference regime.

In the moderately strong interference regime, where  $SNR_2 \le INR_2 \le INR_2^{\dagger}$ , the achievability of (43) is directly from the general achievability region. Note that, in this regime, the rate region is achieved by a mixed scheme, which include both decode-and-forward and quantize-and-forward strategies.

Finally, consider the strong interference regime, where  $INR_2 \ge INR_2^{\dagger}$  and the very strong interference regime, where  $INR_2 \ge INR_2^{\ddagger}$ . We show that (49) and (51) are the capacity regions, respectively.

First, by setting<sup>1</sup>  $R_b = R_0$ ,  $R_a = 0$  and  $\beta = 0$ , the achievable rate region  $\mathcal{R}_{\alpha,\beta}(R_a, R_b)$  in (65) reduces to

$$\begin{cases} (R_1, R_2) & \begin{array}{ccc} R_1 & \leq & \gamma(\mathsf{SNR}_1) \\ R_2 & \leq & \min\{\gamma(\mathsf{SNR}_2) + R_0, \gamma(\mathsf{INR}_2)\} \\ R_1 + R_2 & \leq & \gamma(\mathsf{SNR}_1 + \mathsf{INR}_2) \end{cases} \end{cases}$$
(74)

This rate region reduces to (49) in the strong interference regime, because

$$\gamma(\mathsf{SNR}_2) + R_0 \le \gamma(\mathsf{INR}_2) \tag{75}$$

when  $INR_2 \ge INR_2^{\dagger}$ . Thus, (49) is achievable.

Further, in the very strong interference regime, where  $INR_2 \ge INR_2^{\ddagger}$ , the constraint on  $R_1 + R_2$  in (49) becomes redundant. Thus, the rate region reduces to (51).

Next, we give a converse proof to show that (49) and (51) are indeed the capacity regions in the strong and very strong interference regimes, respectively. Following the same idea as in the converse proof of Theorem 1, we show if  $(R_1, R_2)$  is in the achievable rate region for the Type II Gaussian Z-relay-interference channel, i.e.,  $X_1$  can be reliably decoded at  $Y_1$  at rate  $R_1$ , and  $X_2$  can be reliably decoded at  $Y_2$  at rate  $R_2$ , then  $X_2$  must also be decodable at the  $Y_1$ .

First, observe that by the cut-set upper bound [21], reliable decoding of  $X_2$  at  $Y_2$  requires

$$R_2 \le \gamma(\mathsf{SNR}_2) + R_0. \tag{76}$$

To show that  $X_2$  must be decodable at  $Y_1$ , note that after the decoding of  $X_1$  at  $Y_1$ ,  $X_1$  can be subtracted from the received signal to form

$$\tilde{Y}_1 = h_{21}X_2 + Z_1. \tag{77}$$

The capacity of this channel is  $\gamma(INR_2)$ . On the other hand,  $R_2$  is bounded by  $\gamma(SNR_2) + R_0$ , which is less than  $\gamma(INR_2)$ when  $INR_2 \ge INR_2^{\dagger}$ . So,  $X_2$  is always decodable based on  $\tilde{Y}_1$ .

Now, since both  $X_1$  and  $X_2$  are decodable at  $Y_1$  in the strong interference regime, the achievable rate region of the Gaussian Z-relay-interference channel in the strong interference regime must be a subset of the capacity region of a Gaussian multiple-access channel with  $X_1$ ,  $X_2$  as inputs and  $Y_1$  as output, which is

$$\left\{ \begin{array}{c|cc}
R_1 &\leq & \gamma(\mathsf{SNR}_1) \\
R_2 &\leq & \gamma(\mathsf{INR}_2) \\
R_1 + R_2 &\leq & \gamma(\mathsf{SNR}_1 + \mathsf{INR}_2) \\
\end{array} \right\}. \quad (78)$$

<sup>1</sup>The value of  $\alpha$  does not affect  $\mathcal{R}_{\alpha,\beta}(R_a, R_b)$  when  $R_a = 0$ .

Combining (76), (78), and observing that

$$\gamma(\mathsf{SNR}_2) + R_0 \le \gamma(\mathsf{INR}_2) \tag{79}$$

when  $INR_2 \ge INR_2^{\dagger}$ , we proved that the achievable rate region of the Gaussian Z-relay-interference channel must be bounded by (49) when  $INR_2 \ge INR_2^{\dagger}$ , which reduces to (51) when  $INR_2 \ge INR_2^{\dagger}$ .

The achievability proof for the weak interference case in Theorem 3 is based on a quantize-and-forward scheme in which  $Y_1$  quantizes a fictitious signal  $\bar{Y}_1 = h_{21}(U_2 + W_2) + \alpha^* h_{21}W_2 + Z_1$ , which is a linear combination of its own received signal and the decoded  $W_2$ , with  $\alpha^*$  optimized for maximum overall achievable rate region. The quantized  $\bar{Y}_1$ is decoded with  $Y_2 = h_{22}(U_2 + W_2) + Z_2$  as decoder side information in establishing the multiple-access relay channel capacity region  $C_2$  with Wyner-Ziv coding.

As mentioned earlier, another possible quantize-and-forward strategy is to restrict the decoding order for the multiple-access channel  $C_2$  to be that of decoding  $W_2$  first, then  $U_2$ . In this case,  $W_2$  would be known at *both* the input and the output of the relay link when decoding  $U_2$ . Thus, the relay only has to quantize  $h_{21}U_2 + Z_1$  with  $h_{22}U_2 + Z_2$  as decoder side information in Wyner-Ziv coding.

Surprisingly, these two different strategies give the exact same achievable rate region in the weak-interference regime. This fact is shown in Appendix E, which gives an alternative proof for the weak-interference result in Theorem 3.

## B. Comments on Theorem 3

Unlike the Type I channel, the achievable rate region for the Type II Gaussian Z-relay-interference channel has a more complicated structure. The rate region is no longer neatly divided into weak, strong and very strong interference regimes. There is a new "moderately strong" interference regime, where a combination of decode-and-forward and quantize-andforward strategies may be needed.

In the strong and very strong interference regimes, the relay expands the capacity region by decoding  $X_2$  and forwarding its bin index to help  $Y_2$  decode  $X_2$ . The boundaries of the strong and very strong regimes depend on the relay link rate. In these two regimes, the entire  $X_2$  is common information. Due to the strong interference link, this common message  $X_2$ is guaranteed to be decodable at  $Y_1$ .

As a numerical example, Fig. 12 shows how the capacity region of a Gaussian Z-interference channel in the strong and very strong interference regimes is expanded by the digital relay link from the interfered receiver to the interference-free receiver. The channel parameters are set to be  $SNR_1 = SNR_2 = 20$  dB and  $INR_2 = 55$  dB. Without the digital link, this is a classic Gaussian Z-interference channel in the very strong interference regime, where  $INR_2 \ge SNR_2(1 + SNR_1)$ . In this regime, the capacity region is a rectangle [7], as depicted by the dash-dotted region in Fig.12. With a 2-bit digital link, the Gaussian Z-interference channel remains in the very strong interference regime. The capacity region, given by (51) is depicted by the dashed



Fig. 12. Capacity region of the Gaussian Z-interference channel in the strong interference regime with and without a digital relay link of Type II.



Fig. 13. Capacity region of the Gaussian Z-interference channel in the weak interference regime with and without a digital relay link of Type II.

rectangular region in Fig. 12. It represents a rate increase in  $R_2$  of exactly 2 bits. When  $R_0 = 4$  bits, the Gaussian Zinterference channel falls into the strong interference regime. The capacity region, given by (49), becomes the solid pentagon region. In this regime, for some  $R_1$  lower than a certain threshold, the rate increase in  $R_2$  is exactly 4 bits. For other values of  $R_1$ , the rate increase in  $R_2$  is less than 4 bits. Further increase in the rate of the digital link would increase the rate constraint on  $R_2$ , but not the sum rate.

In the weak interference regime, where  $INR_2 \leq SNR_2$ , Theorem 3 shows that a pure quantize-and-forward of the private message should be used for relaying. Intuitively, this is because, when the interference link is weak, the common message rate is limited by the interference link, which cannot be helped by relaying. Thus, the digital link ought to focus on helping the decoding of private information at  $Y_2$  by quantizeand-forward.

As a numerical example, Fig. 13 shows the achiev-

able rate region of a Gaussian Z-interference channel with  $SNR_1 = SNR_2 = 20$  dB and  $INR_2 = 15$  dB with and without the relay link, which falls into the weak interference regime. The dashed region denoted by points A' and B represents the rate region achieved without the digital link. The solid rate region denoted by points A and B is with a 2-bit digital link.

In Fig. 13, points A and A' both correspond to  $\beta = 1$ , where the entire  $X_2$  is the private message. As  $\beta$  decreases from 1 to 0, the rate pair moves from point A (or A') to point B, which corresponds to  $\beta = 0$ , where the entire  $X_2$ is common message. From the rate pair expression (40), the effect of the digital link is to shift the rate region of the channel without the relay upward by  $\delta(\beta, R_0)$  bits. Since  $\delta(\beta, R_0)$  is monotonically decreasing as  $\beta$  decreases from 1 to 0, for fixed  $R_1$ , the largest increase in  $R_2$  corresponds to  $\delta(1, R_0)$ , i.e. the increase from point A' to A. Note that A and A' are the maximum sum-rate points of the Type II Gaussian Z-interference channel with and without the relay. They correspond to all-private message transmission, which is in contrast to the Type I case where the maximum sum rate is achieved with some  $\beta^* \neq 1$ . Finally, we note that the relay does not affect point B, which corresponds to  $\beta = 0$ , because  $\delta(0, R_0) = 0.$ 

#### C. Sum Capacity Upper Bound

For the Type I channel, the relay link asymptotically achieves the sum capacity cut-set bound in the weak interference regime and high SNR/INR limit. For the Type II channel, however, the cut-set bound is not achievable.

By Theorem 3, an achievable sum rate of the Type II Gaussian Z-interference channel with a relay link of rate  $R_0$  in the weak interference regime is

$$R_{sum} = \gamma \left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + \gamma(\mathsf{SNR}_2) + \delta(1, R_0) \tag{80}$$

which is obtained by setting  $\beta = 1$  in (40). Comparing with the sum capacity of the Gaussian Z-interference channel without the relay in the weak interference regime (23), the sum-rate increase using the relay scheme of Theorem 3 is upper bounded by

$$\delta(1, R_0) = \frac{1}{2} \log \left( \frac{1 + \mathsf{SNR}_2 + \mathsf{INR}_2}{1 + \mathsf{SNR}_2 + 2^{-2R_0} \mathsf{INR}_2} \right)$$
  
$$\leq \gamma \left( \frac{\mathsf{INR}_2}{1 + \mathsf{SNR}_2} \right)$$
  
$$\leq \frac{1}{2}$$
(81)

where  $INR_2 \leq SNR_2$  is used in the last inequality. This is illustrated by an example in Fig. 13, where the rate increase from point A' to point A is about 0.2 bits, which is less than 1/2 bits and is a fraction of the 2-bit capacity of the digital link. This is in stark contrast to the Type I channel, where each bit of the relay capacity can increase the overall sum rate by one bit in the high INR/SNR limit. The half bit upper bound is in fact general, as shown in the following theorem.

Theorem 4: Let  $C(R_0)$  denote the capacity region of the Type II Gaussian Z-interference channel with a digital realy

link of limited rate  $R_0$  from the interfered receiver to the interference-free receiver as shown in Fig. 1(b). Let

$$C_{sum}^{\theta}(R_0) = \max_{(C_1, C_2) \in \mathcal{C}(R_0)} \theta C_1 + (1 - \theta) C_2.$$
(82)

Consider the weak interference regime, where  $INR_2 \leq SNR_2$ . For any  $R_0 \ge 0$ , a capacity upper bound for the Gaussian Z-relay-interference channel is

$$C_{sum}^{\theta}(R_0) \le C_{sum}^{\theta}(0) + (1-\theta)\gamma\left(\frac{\mathsf{INR}_2}{1+\mathsf{SNR}_2}\right).$$
(83)

In particular, as  $R_0 \rightarrow \infty$ , the asymptotic sum capacity of the Type II Gaussian Z-interference channel with a relay link is

$$\gamma\left(\frac{\mathsf{SNR}_1}{1+\mathsf{INR}_2}\right) + \gamma\left(1+\mathsf{INR}_2+\mathsf{SNR}_2\right). \tag{84}$$

Finally, the sum capacity of the Type II Gaussian Z-relayinterference channel is upper bounded by the sum capacity of the Gaussian Z-interference channel without a relay link plus half a bit.

*Proof:* Clearly, the weighted sum capacity gain is a nondecreasing function of  $R_0$ . Thus, we focus on the case of  $R_0 = \infty$ , i.e. when  $Y_2$  has complete knowledge of  $Y_1$ . The proof starts with Fano's inequality:

$$n(\theta R_{1} + (1 - \theta)R_{2}) \\ \stackrel{(a)}{\leq} \quad \theta I(X_{1}^{n};Y_{1}^{n}) + (1 - \theta)I(X_{2}^{n};Y_{1}^{n},Y_{2}^{n}) + n\epsilon_{n} \\ = \quad \theta I(X_{1}^{n};Y_{1}^{n}) + (1 - \theta)I(X_{2}^{n};Y_{2}^{n}) + (1 - \theta)I(X_{2}^{n};Y_{1}^{n}|Y_{2}^{n}) + n\epsilon_{n} \\ \stackrel{(b)}{\leq} \quad nC_{sum}^{\theta}(0) + (1 - \theta)I(X_{2}^{n};Y_{1}^{n}|Y_{2}^{n}) + n\epsilon_{n} \\ \stackrel{(c)}{\leq} \quad nC_{sum}^{\theta}(0) + (1 - \theta)I(X_{2}^{n};Y_{1}^{n}|X_{1}^{n},Y_{2}^{n}) + n\epsilon_{n}$$

where

- follows from Fano's inequality; (a)
- follows from the definition of  $C_{sum}^{\theta}(0)$ ; (b)
- follows from the fact that  $X_1^n$  is independent of  $X_2^n$ (c) given  $Y_2^n$ , in which case

$$\begin{split} I(X_{2}^{n};Y_{1}^{n}|Y_{2}^{n}) &= h(X_{2}^{n}|Y_{2}^{n}) - h(X_{2}^{n}|Y_{1}^{n},Y_{2}^{n}) \\ &\leq h(X_{2}^{n}|Y_{2}^{n}) - h(X_{2}^{n}|X_{1}^{n},Y_{1}^{n},Y_{2}^{n}) \\ &= h(X_{2}^{n}|X_{1}^{n},Y_{2}^{n}) - h(X_{2}^{n}|X_{1}^{n},Y_{1}^{n},Y_{2}^{n}) \\ &= I(X_{2}^{n};Y_{1}^{n}|X_{1}^{n},Y_{2}^{n}). \end{split}$$
(86)

To complete the proof, we only need to show that  $I(X_2^n; Y_1^n | X_1^n, Y_2^n) \le n\gamma \left(\frac{\mathsf{INR}_2}{\mathsf{1}+\mathsf{SNR}_2}\right).$ 

Let  $\Sigma$  denote the covariance matrix of  $X_2^n$ . Define  $\sigma_1^2 =$ 

 $\frac{N}{|h_{21}|^2}$ ,  $\sigma_2^2 = \frac{N}{|h_{22}|^2}$ . Let I be the  $n \times n$  identity matrix. Then,  $I(X_{2}^{n};Y_{1}^{n}|X_{1}^{n},Y_{2}^{n})$  $= h(Y_1^n | X_1^n, Y_2^n) - h(Y_1^n | X_1^n, X_2^n, Y_2^n)$  $h(Y_1^n, Y_2^n | X_1^n) - h(Y_2^n | X_1^n) - h(Z_1^n)$  $= h\left(\left[\begin{array}{c}h_{11}\mathbf{I}\\h_{22}\mathbf{I}\end{array}\right]X_2^n + \left[\begin{array}{c}Z_1\\Z_2^n\end{array}\right]\right) - h\left(h_{22}X_2^n + Z_2^n\right) - h(Z_1^n)$   $\stackrel{(d)}{\leq} \max_{\mathrm{tr}(\mathbf{\Sigma}) \le nP_2} \frac{1}{2}\log\left|\frac{\mathbf{\Sigma} + \sigma_1^2\mathbf{I} - \mathbf{\Sigma}}{|\mathbf{\Sigma} + \sigma_2^2\mathbf{I}| \cdot |\sigma_1^2\mathbf{I}|}\right|$  $= \max_{\operatorname{tr}(\boldsymbol{\Sigma}) \leq nP_2} \frac{1}{2} \log \frac{|\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I}| \cdot |\boldsymbol{\Sigma} + \sigma_1^2 \mathbf{I} - \boldsymbol{\Sigma}(\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I})^{-1} \boldsymbol{\Sigma}|}{|\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I}| \cdot |\sigma_1^2 \mathbf{I}|}$  $\max_{\operatorname{tr}(\boldsymbol{\Sigma}) \leq nP_2} \frac{1}{2} \log \frac{|(\boldsymbol{\Sigma}^{-1} + (\sigma_2^2)^{-1} \mathbf{I})^{-1} + \sigma_1^2 \mathbf{I}|}{|\sigma_1^2 \mathbf{I}|}$  $\max_{\mathbf{I}} \frac{1}{\log} \frac{|\sigma_2^2 \mathbf{I} - \sigma_2^2 \mathbf{I} (\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I})^{-1} \sigma_2^2 \mathbf{I} + \sigma_1^2 \mathbf{I}|}{|\sigma_2^2 \mathbf{I} - \sigma_2^2 \mathbf{I}$ 

$$\underset{=}{\overset{\operatorname{tr}(\mathbf{\Sigma}) \leq nP_2}{=} 2 \operatorname{RS}} \quad |\sigma_1^2 \mathbf{I}|$$

$$\overset{(e)}{=} \quad n\gamma \left( \frac{\operatorname{INR}_2}{1 + \operatorname{SNR}_2} \right)$$

$$(87)$$

where

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- follows from the extremal entropy inequality of Liu (d) and Viswanath [18], which shows that a Gaussian  $X_2^n$  maximizes the entropy difference term immediately above (a detailed proof is given in Appendix F);
- follows from the fact that the maximizing  $\Sigma$  is  $P_2$ **I**, (e) which is shown explicitly in Appendix G;

and the intermediate steps make repeated use of the fact that  $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - BD^{-1}C|$  and the matrix inversion lemma  $(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + C)^{-1}$  $A \quad B$  $DA^{-1}B)^{-1}DA^{-1}$ . Combining (85), (87) and noting that  $\epsilon_n \to 0$  as  $n \to \infty$  give us the desired result (83).

Next, note that the sum capacity of the Type II Gaussian Z-relay-interference channel with a relay link of capacity  $R_0$ can be expressed as  $2C_{sum}^{\frac{1}{2}}(R_0)$ . Further, as given by (23),

$$2C_{sum}^{\frac{1}{2}}(0) = \gamma \left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + \gamma(\mathsf{SNR}_2). \tag{88}$$

Thus, by (83),

$$\gamma \left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + \gamma(\mathsf{SNR}_2) + \gamma \left(\frac{\mathsf{INR}_2}{1 + \mathsf{SNR}_2}\right)$$
$$= \gamma \left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right) + \gamma \left(1 + \mathsf{INR}_2 + \mathsf{SNR}_2\right) \quad (89)$$

is a sum capacity upper bound for  $2C_{sum}^{\frac{1}{2}}(R_0)$ . By Theorem 3 and the computation earlier (i.e. (80)-(81)), we see that the above upper bound is also asymptotically achievable when  $R_0 \rightarrow \infty$ . This proves that (84) is the asymptotic sum capacity. Finally, the half-bit bound on sum capacity follows from the fact that  $\gamma\left(\frac{\mathsf{INR}_2}{1+\mathsf{SNR}_2}\right) \leq \frac{1}{2}$  when  $\mathsf{INR}_2 \leq \mathsf{SNR}_2$ .

Note that the capacity region of the Gaussian Z-interference without the relay link is not yet completely known, except for the sum capacity point. Theorem 4 bounds the capacity



Fig. 14. Gaussian interference channel with a relay link.

increase due to the relay link, without explicitly finding either capacity regions.

The asymptotic sum capacity (84) is essentially the sum capacity of a degraded Gaussian interference channel where the inputs are  $X_1$  and  $X_2$ , and outputs are  $Y_1$  and  $(Y_1, Y_2)$  of a Gaussian Z-interference channel in the weak interference regime. The capacity region for the general degraded interference channel is still open.

#### IV. SUMMARY AND CONCLUDING REMARKS

This paper studies a class of Gaussian Z-interference channels with receiver cooperation, where a rate-limited digital link is provided from one receiver to the other one. It is shown that, when the relay link goes from the interference-free receiver to the interfered receiver, it can significantly enlarge the achievable rate region of a Gaussian Z-interference channel. In this case, the interference-free receiver may decode then binand-forward a part of the interference to the interfered receiver for interference subtraction. This paper shows that, in the strong interference regime, this partial interference forwarding strategy is capacity achieving. In the weak interference regime, the asymptotic sum capacity can be achieved with either a partial-interference-forwarding or a quantize-and-forward strategy in the high SNR/INR limit.

When the digital link is from the interfered receiver to the interference-free receiver, the paper shows that a decode-andforward scheme is capacity achieving in the strong interference regime. In the weak interference regime, an achievable rate region is derived based a quantize-and-forward strategy. In the moderately strong interference regime, a combination of decode-and-forward and quantize-and-forward can be used.

It is interesting to note that in the weak interference regime, when the relay link goes from the interference-free receiver to the interfered receiver, the digital relay essentially achieves the cut-set bound in the high SNR/INR limit—every bit of relay capacity results in one bit increase in sum capacity. However, when the direction of the relay is reversed, the sum capacity increase is upper bounded by half a bit, regardless of the relay link capacity. Thus, a relay link from the interference-free receiver to the interfered receiver is much more efficient.

Our study of the Z-interference channel provides insights into the achievable rates and relay strategies for the general Gaussian interference with a relay link, as shown in Fig. 14. In this more general setting, receiver 1 decodes both users' common messages and observes a noisy version of user 2's



Fig. 15. Gaussian multiple-access channel with a digital relay link.



Fig. 16. Capacity region of the multiple-access channel with a relay link.

private message. This more general setting can be treated as a combination of the two types of Gaussian Z-relay-interference channel considered in this paper. More specifically, receiver 1 may describe both common messages from user 1 and user 2 using bin indices, and describe user 2's private message using a quantize-and-forward strategy. The bin index for user 1's common message mitigates the interference at receiver 2; the bin index for user 2's private message and the quantization of user 2's private message enhances the decoding of user 2's message at receiver 2. In the weak interference regime, which is of most practical interests, the binning of user 1's common message for interference subtraction is expected to be much more efficient than the other two relay modes for enlarging the overall achievable rate region.

#### APPENDIX

#### A. Gaussian Relay Multiple Access Channel

In this appendix, we prove the capacity of a Gaussian multiple-access channel with a relay, where the relay knows one of the inputs perfectly. The channel model is as shown in Fig. 15.

Lemma 1: The capacity region of a Gaussian multipleaccess relay channel  $Y = h_1X_1 + h_2X_2 + Z$ , where  $Z \sim \mathcal{N}(0, N)$  and  $X_1, X_2$  have power constraints  $P_1$  and  $P_2$ , and where a relay observes  $X_1$  perfectly and has a digital link to Y with limited rate  $R_0$ , is

$$\begin{cases} (R_1, R_2) & R_1 \leq \gamma \left(\frac{|h_1|^2 P_1}{N}\right) + R_0 \\ R_2 \leq \gamma \left(\frac{|h_2|^2 P_2}{N}\right) \\ R_1 + R_2 \leq \gamma \left(\frac{|h_1|^2 P_1 + |h_2|^2 P_2}{N}\right) + R_0 \end{cases} \end{cases}$$
(90)

*Proof:* We first prove the achievability. Without the relay link, the capacity of the classic multiple-access channel is a pentagon with corner points A and B, as depicted in Fig. 16

$$\begin{cases} R_1^A = I(X_1; Y | X_2) \\ R_2^A = I(X_2; Y) \end{cases} \begin{cases} R_1^B = I(X_1; Y) \\ R_2^B = I(X_2; Y | X_1). \end{cases}$$
(91)

Points A and B can be achieved by successive decoding with orders  $2 \rightarrow 1$  and  $1 \rightarrow 2$  respectively.

Consider now the multiple-access channel with a relay, where the relay observes  $X_1$  perfectly and where the relay has a rate-limited digital link to Y. Consider first a decoding order  $2 \rightarrow 1$ . The rate achievable by user 2 is still  $I(X_2; Y)$ . With  $X_2$  known at Y, consider now the operation of the relay. Since the relay has perfect knowledge of  $X_1$ , the channel from  $X_1$  to Y is a degraded channel [9]. Thus, the following rate pair is achievable:

$$R_1^{A'} = I(X_1; Y|X_2) + R_0 = R_1^A + R_0$$
 (92)

$$R_2^{A'} = I(X_2; Y) = R_2^A.$$
(93)

Similarly, for the decoding order  $1 \rightarrow 2$ , the following rate pair is achievable:

$$R_1^{B'} = I(X_1; Y) + R_0 = R_1^B + R_0$$
(94)

$$R_2^{B'} = I(X_2; Y | X_1) = R_2^B.$$
(95)

With time sharing, we arrive at the pentagon region with corner points A' and B' in Fig. 16. Applying independent Gaussian input distributions  $X_1 \sim \mathcal{N}(0, P_1)$  and  $X_2 \sim \mathcal{N}(0, P_2)$ , we prove the achievability of (90).

The converse follows from the cut-set upper bound [21]. Let  $\tilde{V}^n$  and  $V^n$  be the input and the output of the digital relay link, respectively, taking values in  $\{1, 2, \dots, 2^{nR_0}\}$ . Then,

$$nR_{1} \leq I(X_{1}^{n}, \tilde{V}^{n}; Y^{n}, V^{n} | X_{2}^{n})$$

$$= I(X_{1}^{n}; Y^{n}, V^{n} | X_{2}^{n}) + I(\tilde{V}^{n}; Y^{n}, V^{n} | X_{1}^{n}, X_{2}^{n})$$

$$= I(X_{1}^{n}; Y^{n} | X_{2}^{n}) + I(X_{1}^{n}; V^{n} | X_{2}^{n}, Y^{n}) +$$

$$H(\tilde{V}^{n} | X_{1}^{n}, X_{2}^{n}) - H(\tilde{V}^{n} | X_{1}^{n}, X_{2}^{n}, Y^{n}, V^{n})$$

$$\stackrel{(a)}{=} I(X_{1}^{n}; Y^{n} | X_{2}^{n}) + I(X_{1}^{n}; V^{n} | X_{2}^{n}, Y^{n})$$

$$\stackrel{(b)}{\leq} I(X_{1}^{n}; Y^{n} | X_{2}^{n}) + nR_{0}$$
(96)

where (a) comes from the fact that  $\tilde{V}^n$  is a function of  $X_1^n$ , so

$$H(\tilde{V}^n|X_1^n, X_2^n) = H(\tilde{V}^n|X_1^n, X_2^n, Y^n, V^n), \quad (97)$$

and (b) comes from the following inequality

$$I(X_{1}^{n}; V^{n} | X_{2}^{n}, Y^{n}) = H(V^{n} | X_{2}^{n}, Y^{n}) - H(V^{n} | X_{1}^{n}, X_{2}^{n}, Y^{n}) \\ \leq H(V^{n} | X_{2}^{n}, Y^{n}) \\ \leq nR_{0}.$$
(98)

Proceeding with smilar lines of reasoning, we can also prove that

$$nR_2 \leq I(X_2^n; Y^n | X_1^n) \tag{99}$$

$$nR_1 + nR_2 \leq I(X_1^n, X_2^n; Y^n) + nR_0.$$
 (100)

The above constraints on  $R_1$ ,  $R_2$  and  $R_1 + R_2$  are simultaneously maximized by i.i.d. Gaussian inputs  $X_1^n$  distributed as



Fig. 17. The region defined by lines  $R_1 = \gamma(SNR_1)$ ,  $R_2 = \gamma(SNR_2) + R_0$  and the curve (101).

 $\mathcal{N}(0, P_1)$  and  $X_2^n$  distributed as  $\mathcal{N}(0, P_2)$ . The converse proof is completed by evaluating the mutual information expression in (96), (99) and (100) with Gaussian inputs.

#### B. Convexity of Achievable Rate Region (14)

This appendix shows that the region defined by  $R_1 \leq \gamma(SNR_1)$ ,  $R_2 \leq \gamma(SNR_2) + R_0$ , and the curve

$$\begin{cases} R_1 = \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right) \\ R_2 = \gamma (\beta \mathsf{SNR}_2) + \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + R_0 \\ (101) \end{cases}$$

where  $0 \le \beta \le 1$ , is convex when  $INR_2 \le SNR_2$ .

Note that, when  $\beta = 1$  and  $\beta = 0$ , the curve defined by (101) meets  $R_2 = \gamma(SNR_2) + R_0$  and  $R_1 = \gamma(SNR_1)$  at points A and B, respectively, as shown in Fig. 17. Therefore, to prove the convexity of the region, we only need to prove that the curve (101) parameterized by  $\beta$  is concave.

First, we express  $\beta$  in terms of  $R_1$ :

ν

$$\beta = \frac{1}{\mathsf{INR}_2} \left( \frac{\mathsf{SNR}_1}{2^{2R_1} - 1} - 1 \right). \tag{102}$$

Substituting this expression for  $\beta$  into the expression for  $R_2$  in (101), we obtain  $R_2$  as a function of  $R_1$ :

$$R_2 = \frac{1}{2} \log \left( -\nu 2^{2R_1} + \lambda \right) + \mu \tag{103}$$

where

$$= \frac{\mathsf{SNR}_2}{\mathsf{INR}_2} - 1 \tag{104}$$

$$\lambda = \frac{\mathsf{SNR}_2}{\mathsf{INR}_2}(1+\mathsf{SNR}_1) - 1 \tag{105}$$

$$\mu = \gamma \left(\frac{1 + \mathsf{INR}_2}{\mathsf{SNR}_1}\right) + R_0. \tag{106}$$

Note that when  $INR_2 \leq SNR_2$ ,  $\nu \geq 0$  and  $\lambda > 0$ .

Observe that  $R_1$  is a monotonic decreasing function of  $\beta$ . So, in the range  $0 \le \beta \le 1$ , we have

$$\gamma\left(\frac{\mathsf{SNR}_1}{1+\mathsf{INR}_2}\right) \le R_1 \le \gamma(\mathsf{SNR}_1).$$
 (107)

In this range of  $R_1$ , it is easy to verify that  $-\nu 2^{2R_1} + \lambda > 0$ .

Now, taking the first and second order derivatives of  $R_2$  another term  $h(\hat{Y}_2|Y_1, X_1)$ : with respect to  $R_1$  in (103), we have

$$R_2' = \frac{-\nu 2^{2R_1}}{-\nu 2^{2R_1} + \lambda} \tag{108}$$

$$R_2'' = \frac{-2\lambda\nu 2^{2R_1}}{(-\nu 2^{2R_1} + \lambda)^2} \ln 2.$$
 (109)

Since  $\nu \ge 0$ ,  $\lambda > 0$ , and  $-\nu 2^{2R_1} + \lambda > 0$ , we have  $R'_2 \le 0$ and  $R''_2 \le 0$ . As a result, the curve (101) parameterized by  $\beta$ is concave. Therefore, in the weak interference regime where  $\mathsf{INR}_2 \le \mathsf{SNR}_2$ , the rate region define by  $R_1 \le \gamma(\mathsf{SNR}_1)$ ,  $R_2 \le \gamma(\mathsf{SNR}_2) + R_0$  and (101) is convex.

#### C. Evaluation of Wyner-Ziv Rate (38)

In this appendix, we show that with independent Gaussian inputs  $X_1 \sim \mathcal{N}(0, P_1)$  and  $X_2 \sim \mathcal{N}(0, P_2)$ , and the Gaussian quantization scheme (37), the achievable rate described by (35), (34) and (36) is given by (38).

Clearly, with a Gaussian input  $X_2 \sim \mathcal{N}(0, P_2)$ ,  $R_2$  is given by

$$R_2 = I(X_2; Y_2)$$
  
=  $\gamma(\mathsf{SNR}_2).$  (110)

To fully utilize the digital link, we set  $\hat{Y}_2$  to be such that  $I(Y_2; \hat{Y}_2|Y_1)$  is equal to  $R_0$  in (36). Note that  $\hat{Y}_2 = Y_2 + e$ , where  $Y_2$  and e are independent and  $e \sim \mathcal{N}(0, \sigma^2)$ . To find  $\sigma^2$ , note that

$$R_{0} = I(Y_{2}; \hat{Y}_{2}|Y_{1})$$

$$= h(\hat{Y}_{2}|Y_{1}) - h(\hat{Y}_{2}|Y_{1}, Y_{2})$$

$$= h(Y_{2} + e|Y_{1}) - \frac{1}{2}\log(2\pi e\sigma^{2})$$

$$= \frac{1}{2}\log\left(2\pi e(\sigma^{2}_{Y_{2}|Y_{1}} + \sigma^{2})\right) - \frac{1}{2}\log(2\pi e\sigma^{2})$$

$$= \frac{1}{2}\log\left(1 + \frac{\sigma^{2}_{Y_{2}|Y_{1}}}{\sigma^{2}}\right)$$
(111)

where  $\sigma_{Y_2|Y_1}^2$ , the conditional minimum mean-squared error of  $Y_2$  given  $Y_1$ , can be calculated as follows:

$$\sigma_{Y_2|Y_1}^2 = N + \frac{|h_{22}|^2 P_2(|h_{11}|^2 P_1 + N)}{|h_{11}|^2 P_1 + |h_{21}|^2 P_2 + N}$$
  
=  $N\left(1 + \frac{\mathsf{SNR}_2(1 + \mathsf{SNR}_1)}{1 + \mathsf{SNR}_1 + \mathsf{INR}_2}\right).$  (112)

Substituting the above  $\sigma_{Y_2|Y_1}^2$  into (111), we have

$$\sigma^{2} = \frac{\sigma_{Y_{2}|Y_{1}}^{2}}{2^{2R_{0}} - 1}$$
$$= \frac{N}{2^{2R_{0}} - 1} \left( 1 + \frac{\mathsf{SNR}_{2}(1 + \mathsf{SNR}_{1})}{1 + \mathsf{SNR}_{1} + \mathsf{INR}_{2}} \right). \quad (113)$$

Now, we are ready to calculate  $R_1$ . First, we evaluate

$$h(\tilde{Y}_{2}|Y_{1}, X_{1}) = h(h_{22}X_{2} + Z_{2} + e|h_{11}X_{1} + h_{21}X_{2} + Z_{1}, X_{1})$$
  
$$= h(h_{22}X_{2} + Z_{2} + e|h_{21}X_{2} + Z_{1})$$
  
$$= \frac{1}{2}\log\left(2\pi e\left(\sigma^{2} + N + \frac{|h_{22}|^{2}P_{2}N}{|h_{21}|^{2}P_{2} + N}\right)\right)$$
  
$$= \frac{1}{2}\log\left(2\pi e\left(\sigma^{2} + N\left(1 + \frac{\mathsf{SNR}_{2}}{1 + \mathsf{INR}_{2}}\right)\right)\right) (114)$$

where  $\sigma^2$  is given by (113).

Now, the rate of user 1 is given by

$$R_{1} = I(X_{1}; Y_{1}, Y_{2})$$
  
=  $I(X_{1}; Y_{1}) + I(X_{1}; \hat{Y}_{2}|Y_{1})$   
=  $I(X_{1}; Y_{1}) + h(\hat{Y}_{2}|Y_{1}) - h(\hat{Y}_{2}|Y_{1}, X_{1}).$  (115)

Clearly, with independent Gaussian inputs  $X_1 \sim \mathcal{N}(0, P_1)$ and  $X_2 \sim \mathcal{N}(0, P_2)$ ,

$$I(X_1; Y_1) = \gamma \left(\frac{\mathsf{SNR}_1}{1 + \mathsf{INR}_2}\right). \tag{116}$$

Substituting (116), (114) and  $h(\hat{Y}_2|Y_1)$  from (111) into (115), after some calculations, we obtain  $R_1$  in (38).

# D. Convexity of Achievable Rate Region (73)

This appendix proves that the region defined by  $R_1 \leq \gamma(SNR_1)$ ,  $R_2 \leq \gamma(SNR_2) + \delta(\beta, R_0)$ , and the curve

$$\begin{bmatrix}
R_1 \leq \gamma \left( \frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2} \right) \\
R_2 \leq \gamma (\beta \mathsf{SNR}_2) + \gamma \left( \frac{\overline{\beta} \mathsf{INR}_2}{1 + \mathsf{SNR}_1 + \beta \mathsf{INR}_2} \right) + \delta(\beta, R_0) \\
(117)$$

where  $0 \le \beta \le 1$ , is convex when  $INR_2 \le SNR_2$ .

We follow the same idea used in Appendix B to prove the convexity of the above region. By Appendix B, we can rewrite  $R_2$  as

$$R_{2} = \frac{1}{2} \log \left( -\nu 2^{2R_{1}} + \lambda \right) + \tilde{\mu} + \delta(\beta, R_{0})$$
(118)

where  $\tilde{\mu} = \mu - R_0$  is a constant, and  $\nu, \lambda, \mu$  are as defined in Appendix B.

It is easy to verify that in the weak interference regime,  $\delta(\beta, R_0)$  is concave in  $\beta$ , and  $\beta(R_1)$ , as denoted in (102), is convex in  $R_1$ . Combining this with the fact that  $\delta(\beta, R_0)$  is a nondecreasing function of  $\beta$  shows that  $\delta(\beta, R_0)$  is a concave function of  $R_1$ . Adding  $\delta(\beta, R_0)$  with another concave (proved in Appendix B) term  $\frac{1}{2} \log (-\nu 2^{2R_1} + \lambda) + \tilde{\mu}$  gives us the desired result that  $R_2$  is a concave function of  $R_1$ .

Therefore, the region defined by  $R_1 \leq \gamma(SNR_1)$ ,  $R_2 \leq \gamma(SNR_2) + \delta(\beta, R_0)$  and (117) is convex.

#### E. Alternative Proof of (40)

In this appendix, we give an alternative proof of the achievability region (40) for the Type II channel in the weak

interference regime. Start with the same  $C_1$  as in (52), but fix a decoding order so that

$$\begin{cases} S_1 = I(X_1; Y_1 | W_2) \\ T_2 = I(W_2; Y_1). \end{cases}$$
(119)

Next, we derive a new  $C_2$  with a fixed decoding order of decoding  $W_2$  first, then  $U_2$ , so that

$$\begin{cases} T_2 = I(W_2; Y_2) \\ S_2 = I(U_2; Y_2, \hat{Y}'_1 | W_2) \end{cases}$$
(120)

where  $\hat{Y}_1'$  is the result of quantizing

$$\bar{Y}_1' = h_{21}U_2 + Z_1 \tag{121}$$

with a Gaussian Wyner-Ziv quantizer

$$\hat{Y}_1' = \bar{Y}_1' + e, \tag{122}$$

where  $e \sim \mathcal{N}(0, \sigma^2)$  is independent of  $\bar{Y}_1$ . Note that this quantization scheme is different from the one described in (53). In addition, since  $W_2$  is decoded first at  $Y_2$ , it serves as decoder side information in Wyner-Ziv coding.

To compute the variance of e, note that

$$R_{0} = I(\hat{Y}_{1}'; \bar{Y}_{1}'|Y_{2}, W_{2})$$

$$= h(\hat{Y}_{1}'|Y_{2}, W_{2}) - h(\hat{Y}_{1}'|\bar{Y}_{1}', Y_{2}, W_{2})$$

$$= h(\bar{Y}_{1}' + e|h_{22}U_{2} + Z_{2}) - \frac{1}{2}\log(2\pi e\sigma^{2})$$

$$= \frac{1}{2}\log\left(2\pi e(\sigma_{\bar{Y}_{1}|h_{22}U_{2} + Z_{2}}^{2} + \sigma^{2})\right) - \frac{1}{2}\log\left(2\pi e\sigma^{2}\right)$$

$$= \frac{1}{2}\log\left(1 + \frac{\sigma_{\bar{Y}_{1}|h_{22}U_{2} + Z_{2}}^{2}}{\sigma^{2}}\right)$$
(123)

where  $\sigma^2_{Y_1'|h_{22}U_2+Z_2}$ , the conditional minimum mean-squared error of  $Y_1'$  given  $h_{22}U_2 + Z_2$ , can be calculated as

$$\sigma_{\bar{Y}_{1}^{\prime}|h_{22}U_{2}+Z_{2}}^{2} = N + \frac{\beta |h_{21}|^{2} P_{2} N}{\beta |h_{22}|^{2} P_{2} + N}$$
$$= N \left(1 + \frac{\beta \mathsf{INR}_{2}}{1 + \beta \mathsf{SNR}_{2}}\right) \quad (124)$$

Substituting the above  $\sigma^2_{\tilde{Y}'_1|h_{22}U_2+Z_2}$  into (123), we have

$$\sigma^{2} = \frac{\sigma_{\bar{Y}_{1}|h_{22}U_{2}+Z_{2}}^{2}}{2^{2R_{0}}-1} = \frac{N}{2^{2R_{0}}-1} \left(1 + \frac{\beta \mathsf{INR}_{2}}{1 + \beta \mathsf{SNR}_{2}}\right)$$
(125)

Now, we evaluate the overall achievable rate region by noting that

$$R_1 = S_1 = I(X_1; Y_1 | W_2) = \gamma \left(\frac{\mathsf{SNR}_1}{1 + \beta \mathsf{INR}_2}\right)$$
 (126)

and since  $I(W_2; Y_1) \leq I(W_2, Y_2)$  when  $SNR_2 \geq INR_2$ ,

$$R_{2} = T_{2} + S_{2}$$

$$= I(W_{2}; Y_{1}) + I(U_{2}; Y_{2}, \hat{Y}'_{1}|W_{2})$$

$$= I(W_{2}; Y_{1}) + I(U_{2}; Y_{2}|W_{2}) + I(U_{2}; \hat{Y}'_{1}|Y_{2}, W_{2})$$

$$= \gamma \left(\frac{\overline{\beta}\mathsf{INR}_{2}}{1 + \mathsf{SNR}_{1} + \beta\mathsf{INR}_{2}}\right) + \gamma(\beta\mathsf{SNR}_{2})$$

$$+ I(U_{2}; \hat{Y}'_{1}|Y_{2}, W_{2}). \qquad (127)$$



Fig. 18. Gaussian  $X_2$  maximizes the mutual information difference  $I(X_2; Y_1, Y_2) - I(X_2; \overline{Y}_1, Y_2)$  for any  $\kappa$ , and in particular for  $\kappa \to \infty$ .

To compute the last term in the above,

$$I(U_{2}; \hat{Y}_{1}'|Y_{2}, W_{2}) = \frac{1}{2} \log \left( \frac{\sigma_{\hat{Y}_{1}'|h_{21}U_{2}+Z_{2}}}{\sigma_{\hat{Y}_{1}'|U_{2}}^{2}} \right)$$
$$= \gamma \left( \frac{\beta \mathsf{INR}_{2}}{(1 + \beta \mathsf{SNR}_{2})(1 + \frac{\sigma^{2}}{N})} \right) (128)$$

where the computation of  $\sigma^2_{\hat{Y}'_1|h_{21}U_2+Z_2}$  is similar to (124). Finally, substituting (125) into the above, we obtain

$$R_{2} = \gamma \left( \frac{\overline{\beta} \mathsf{INR}_{2}}{1 + \mathsf{SNR}_{1} + \beta \mathsf{INR}_{2}} \right) + \gamma (\beta \mathsf{SNR}_{2}) + \gamma \left( \frac{\beta (2^{2R_{0}} - 1)\mathsf{INR}_{2}}{2^{2R_{0}}(1 + \beta \mathsf{SNR}_{2}) + \beta \mathsf{INR}_{2}} \right) \quad (129)$$

which is exactly (40).

# F. Proof of Step (d) in (87)

In this appendix, we use the extremal entropy inequality of Liu and Viswanath [18] to show that a Gaussian  $X_2^n$ maximizes the following term

$$h\left(\left[\begin{array}{c}h_{11}\mathbf{I}\\h_{22}\mathbf{I}\end{array}\right]X_2^n + \left[\begin{array}{c}Z_1^n\\Z_2^n\end{array}\right]\right) - h\left(h_{22}X_2^n + Z_2^n\right) - h(Z_1^n).$$
(130)

where  $Z_1^n$  and  $Z_2^n$  are independent.

Redefine  $Y_1 = X_2 + Z_1$ ,  $Y_2 = X_2 + Z_2$ . The above entropy difference is essentially

$$I(X_2^n; Y_1^n, Y_2^n) - I(X_2^n; Y_2^n)$$
(131)

which is the difference in mutual information for a Gaussian vector channel between when receiver knows both  $Y_1$  and  $Y_2$  and when receiver knows only  $Y_2$ .

Gaussian input maximizes individual Gaussian vector channel mutual information. Does Gaussian input also maximize the mutual information gain due to an independent "second look"? We show this is so using a limiting argument.

Let 
$$\overline{Z}_1 = \kappa Z_1$$
 and  $\overline{Y}_1 = X_2 + \overline{Z}_1$  as shown in Fig. 18.

Then,

$$\begin{split} I(X_{2}^{n};Y_{1}^{n},Y_{2}^{n}) &= I(X_{2}^{n};Y_{2}^{n}) \\ &= \lim_{\kappa \to \infty} I(X_{2}^{n};Y_{1}^{n},Y_{2}^{n}) - I(X_{2}^{n};\bar{Y}_{1}^{n},Y_{2}^{n}) \\ &= \lim_{\kappa \to \infty} h(Y_{1}^{n},Y_{2}^{n}) - h(\bar{Y}_{1}^{n},Y_{2}^{n}) - h(Z_{1}^{n}) + h(\bar{Z}_{1}^{n}) \\ &= \lim_{\kappa \to \infty} h\left( \left[ \begin{array}{c} h_{11}\mathbf{I} \\ h_{22}\mathbf{I} \end{array} \right] X_{2}^{n} + \left[ \begin{array}{c} Z_{1}^{n} \\ Z_{2}^{n} \end{array} \right] \right) \\ &-h\left( \left[ \begin{array}{c} h_{11}\mathbf{I} \\ h_{22}\mathbf{I} \end{array} \right] X_{2}^{n} + \left[ \begin{array}{c} \bar{Z}_{1}^{n} \\ Z_{2}^{n} \end{array} \right] \right) \\ &-h(Z_{1}^{n}) + h(\bar{Z}_{1}^{n}) \end{split}$$
(132

We now apply the extremal entropy inequality of Liu and Viswanath [18] to the first two terms in the above to conclude that under a trace constraint  $\operatorname{tr}(\Sigma) \leq nP_2$ , the maximizing  $\begin{bmatrix} h_{11}\mathbf{I} \\ h_{22}\mathbf{I} \end{bmatrix} X_2^n$  must be Gaussian, hence the maximizing  $X_2^n$  must be Gaussian. Consequently, with the entropy expressions evaluated under Gaussian distribution,

$$\begin{split} I(X_{2}^{n};Y_{1}^{n},Y_{2}^{n}) &- I(X_{2}^{n};Y_{2}^{n}) \\ &\leq \lim_{\kappa \to \infty} \max_{\operatorname{tr}(\mathbf{\Sigma}) \leq nP_{2}} \frac{1}{2} \log \frac{\begin{vmatrix} \mathbf{\Sigma} + \sigma_{1}^{2}\mathbf{I} & \mathbf{\Sigma} \\ \mathbf{\Sigma} & \mathbf{\Sigma} + \sigma_{2}^{2}\mathbf{I} \end{vmatrix} \cdot |\kappa^{2}\sigma_{1}^{2}\mathbf{I}|}{|\mathbf{\Sigma} + \kappa^{2}\sigma_{1}^{2}\mathbf{I} & \mathbf{\Sigma} \\ \mathbf{\Sigma} & \mathbf{\Sigma} + \sigma_{2}^{2}\mathbf{I} \end{vmatrix} \cdot |\sigma_{1}^{2}\mathbf{I}|} \\ &= \max_{\operatorname{tr}(\mathbf{\Sigma}) \leq nP_{2}} \frac{1}{2} \log \frac{\begin{vmatrix} \mathbf{\Sigma} + \sigma_{1}^{2}\mathbf{I} & \mathbf{\Sigma} \\ \mathbf{\Sigma} & \mathbf{\Sigma} + \sigma_{2}^{2}\mathbf{I} \end{vmatrix}}{|\mathbf{\Sigma} + \sigma_{2}^{2}\mathbf{I}| \cdot |\sigma_{1}^{2}\mathbf{I}|} + \\ \lim_{\kappa \to \infty} \max_{\operatorname{tr}(\mathbf{\Sigma}) \leq nP_{2}} \frac{1}{2} \log \frac{|\kappa^{2}\sigma_{1}^{2}\mathbf{I}|}{|\mathbf{\Sigma} + \kappa^{2}\sigma_{1}^{2}\mathbf{I} - \mathbf{\Sigma}(\mathbf{\Sigma} + \sigma_{2}^{2}\mathbf{I})^{-1}\mathbf{\Sigma}|} \end{split}$$

$$\leq \max_{\operatorname{tr}(\mathbf{\Sigma}) \leq nP_2} \frac{1}{2} \log \frac{\left| \frac{\boldsymbol{\Sigma} + \boldsymbol{\sigma}_1 \mathbf{I}}{|\mathbf{\Sigma} + \boldsymbol{\sigma}_2^2 \mathbf{I}| \cdot |\boldsymbol{\sigma}_1^2 \mathbf{I}|} \right|}{|\mathbf{\Sigma} + \boldsymbol{\sigma}_2^2 \mathbf{I}| \cdot |\boldsymbol{\sigma}_1^2 \mathbf{I}|} + \lim_{\kappa \to \infty} \max_{\operatorname{tr}(\mathbf{\Sigma}) \leq nP_2} \frac{1}{2} \log \frac{|\kappa^2 \boldsymbol{\sigma}_1^2 \mathbf{I}|}{|\mathbf{\Sigma} + \kappa^2 \boldsymbol{\sigma}_1^2 \mathbf{I}|}$$
(133)

where we again used  $\begin{vmatrix} A & B \\ C & D \end{vmatrix} = |D| \cdot |A - BD^{-1}C|$ . The second term above is clearly zero as  $\kappa \to \infty$ . This proves step (d) of (87).

## G. Maximization in Step (e) of (87)

In this appendix, we explicitly solve

$$\max_{\operatorname{tr}(\boldsymbol{\Sigma}) \le nP_2} \frac{1}{2} \log \frac{|\sigma_2^2 \mathbf{I} - \sigma_2^2 \mathbf{I}(\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I})^{-1} \sigma_2^2 \mathbf{I} + \sigma_1^2 \mathbf{I}|}{|\sigma_1^2 \mathbf{I}|} \quad (134)$$

where  $\sigma_1^2 = \frac{N}{|h_{21}|^2}$ ,  $\sigma_2^2 = \frac{N}{|h_{22}|^2}$ . Over the set of semidefinite matrices, the matrix function

Over the set of semidefinite matrices, the matrix function above is a composition of a concave function with a concave and increasing function, and is therefore concave in  $\Sigma$ . The optimization problem satisfies the Slater's condition; its Karush-Kuhn-Tucker (KKT) condition is therefore necessary and sufficient for optimality. Write down the Lagrangian of the optimization problem

$$\frac{1}{2}\log\frac{|\sigma_2^2\mathbf{I} - \sigma_2^2\mathbf{I}(\boldsymbol{\Sigma} + \sigma_2^2\mathbf{I})^{-1}\sigma_2^2\mathbf{I} + \sigma_1^2\mathbf{I}|}{|\sigma_1^2\mathbf{I}|} - \lambda(\operatorname{tr}(\boldsymbol{\Sigma}) - nP_2)$$
(135)

Taking the derivative of the above matrix function with respect to  $\Sigma$  and setting it to zero, we obtain

$$\frac{1}{2}\sigma_2^2(\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I})^{-1}(\sigma_2^2 \mathbf{I} - \sigma_2^2 \mathbf{I}(\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I})^{-1}\sigma_2^2 \mathbf{I} + \sigma_1^2 \mathbf{I})^{-1}$$
$$(\boldsymbol{\Sigma} + \sigma_2^2 \mathbf{I})^{-1}\sigma_2^2 = \lambda \mathbf{I}. \quad (136)$$

It is easy to verify that a diagonal  $\Sigma = P_2 \mathbf{I}$  along with some positive  $\lambda$  satisfies the above. Therefore, the optimal  $\Sigma$  is  $P_2 \mathbf{I}$ . An evaluation of (134) with a diagonal  $\Sigma = P_2 \mathbf{I}$  then yields its maximum value as  $n\gamma \left(\frac{\mathsf{INR}_2}{\mathsf{I}+\mathsf{SNR}_2}\right)$ .

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