Multirate Digital Signal Processing: Part II

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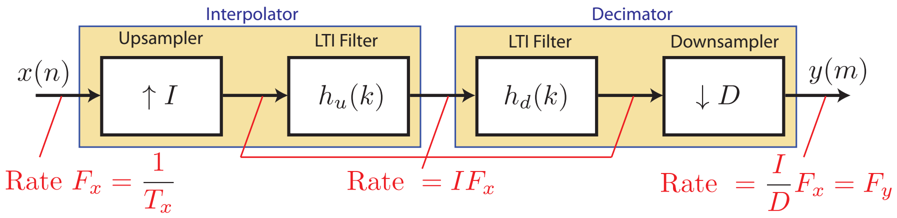
Discrete-Time Signals and Systems

Reference:

Section 11.4 of

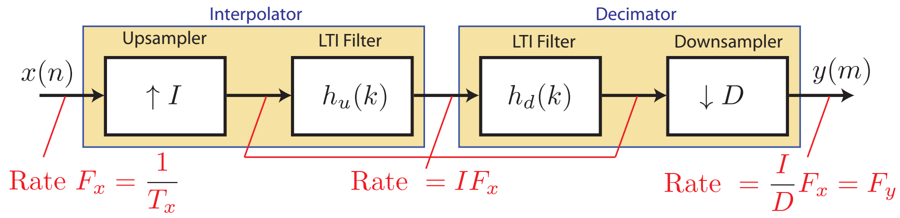
John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.

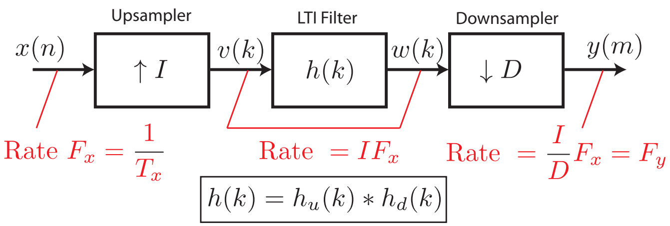
Sampling Rate Conversion by I/D

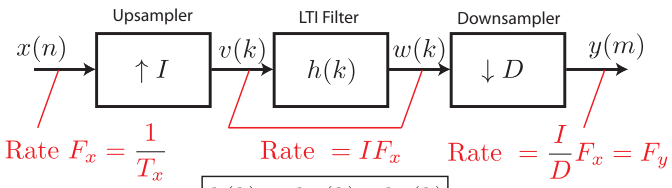


- $x(n)$: original samples at sampling rate F_x
- $y(n)$: new samples at sampling rate F_y

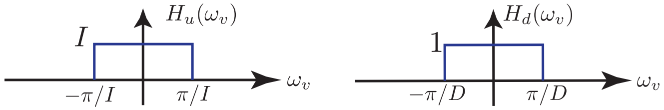
Sampling Rate Conversion by I/D



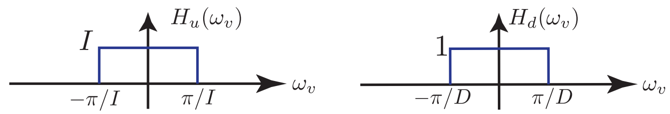




$$\boxed{h(k) = h_u(k) * h_d(k)}$$



Note: $\omega_v = \frac{\omega_x}{I} = \omega_y D$

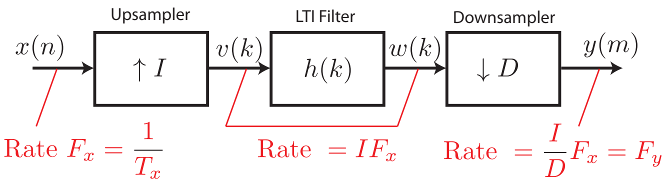


$$H(\omega_v) = H_u(\omega_v)H_d(\omega_v) = \begin{cases} I & 0 \le |\omega_v| \le \min(\pi/D, \pi/I) \\ 0 & \text{otherwise} \end{cases}$$

Note: $\omega_v = \frac{\omega_x}{I}$

Therefore, $h(k) \overset{\mathcal{F}}{\longleftrightarrow} H(\omega_v)$ represents a low pass filter.

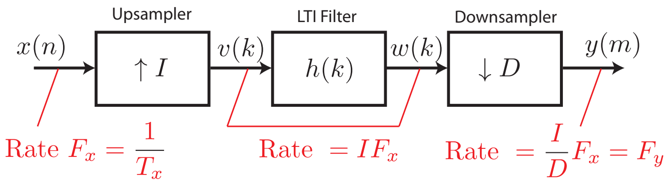
Time Domain Perspective



$$v(k) = \begin{cases} x(k/I) & k = 0, \pm I, \pm 2I, \dots \\ 0 & \text{otherwise} \end{cases}$$

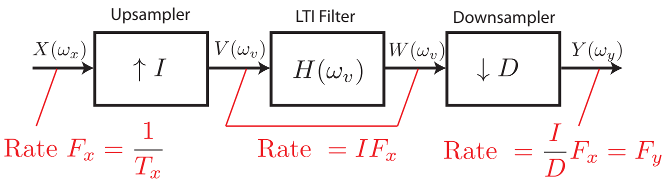
$$\begin{aligned} w(k) &= \sum_{r=-\infty}^{\infty} h(k-r)v(r) \\ &= \cdots + h(k-I)v(I) + h(k)v(0) + h(k+I)v(-I) + \cdots \\ &= \sum_{r=-\infty}^{\infty} h(k-Ir)v(Ir) = \sum_{r=-\infty}^{\infty} h(k-Ir)x(r) \end{aligned}$$

Time Domain Perspective



$$\begin{aligned} w(k) &= \sum_{r=-\infty}^{\infty} h(k-Ir)x(r) \\ y(m) &= w(mD) \\ &= \sum_{r=-\infty}^{\infty} h(mD-Ir)x(r) \\ &= \underbrace{\sum_{r=-\infty}^{\infty} h(mD-Ir)x(r)}_{\text{linear periodically time-varying system}} \end{aligned}$$

Frequency Domain Perspective

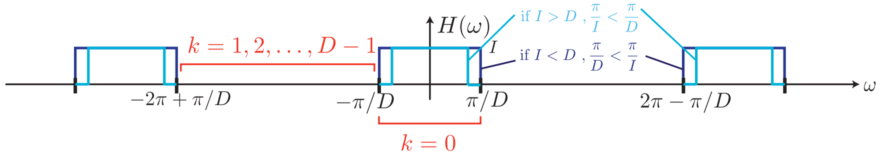


$$
\begin{aligned}
V(\omega_v) &= X(\omega_v I) \quad \text{(for upsampling)}\\
W(\omega_v) &= H(\omega_v)V(\omega_v) = H(\omega_v)X(\omega_v I)\\
Y(\omega_y) &= \frac{1}{D}\sum_{k=0}^{D-1} W\left(\frac{\omega_y - 2\pi k}{D}\right) \quad \text{(for downsampling)}\\
&= \frac{1}{D}\sum_{k=0}^{D-1} H\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{I(\omega_y - 2\pi k)}{D}\right)
\end{aligned}
$$

$$
Y(\omega_y) = \frac{1}{D}\sum_{k=0}^{D-1} H\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{I(\omega_y - 2\pi k)}{D}\right)
$$

For $-\pi \le \omega_y \le \pi$,

$$
\begin{array}{rcll}
-\frac{\pi}{D} \le & \frac{\omega_y - 2\pi k}{D} & \le \frac{\pi}{D} & \text{for } k = 0\\
-\frac{3\pi}{D} \le & \frac{\omega_y - 2\pi k}{D} & \le -\frac{\pi}{D} & \text{for } k = 1\\
& \vdots & \vdots & \\
-2\pi + \frac{\pi}{D} \le & \frac{\omega_y - 2\pi k}{D} & \le -2\pi + \frac{3\pi}{D} & \text{for } k = D-1
\end{array}
$$



Note: For $-\pi \le \omega_y \le \pi$,

$$
H\left(\frac{\omega_y - 2\pi k}{D}\right) = \begin{cases} I \text{ or } 0 & \text{for } k = 0\\ 0 & \text{for } k = 1, 2, \ldots, D-1 \end{cases}
$$

Therefore, for $-\pi \le \omega_y \le \pi$,

$$
\begin{aligned}
Y(\omega_y) &= \frac{1}{D}\sum_{k=0}^{D-1} H\left(\frac{\omega_y - 2\pi k}{D}\right) X\left(\frac{I(\omega_y - 2\pi k)}{D}\right)\\
&= \frac{1}{D}\underbrace{H\left(\frac{\omega_y}{D}\right)}_{= I \text{ or } 0} X\left(\frac{I\omega_y}{D}\right) + \frac{1}{D}\sum_{k=1}^{D-1}\underbrace{H\left(\frac{\omega_y - 2\pi k}{D}\right)}_{=0} X\left(\frac{I(\omega_y - 2\pi k)}{D}\right)
\end{aligned}
$$

$$
H(\omega) = \begin{cases} I & |\omega| < \min(\frac{\pi}{D}, \frac{\pi}{I})\\ 0 & \text{otherwise} \end{cases}
$$

$$
H\left(\frac{\omega}{D}\right) = \begin{cases} I & \left|\frac{\omega}{D}\right| < \min(\frac{\pi}{D}, \frac{\pi}{I})\\ 0 & \text{otherwise} \end{cases} = \begin{cases} I & |\omega| < \min(\pi, \frac{D\pi}{I})\\ 0 & \text{otherwise} \end{cases}
$$

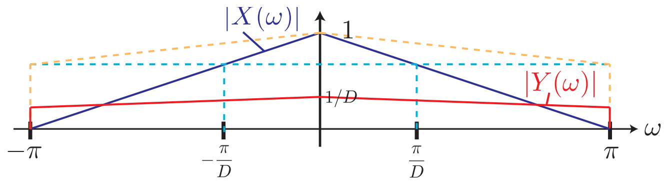
$$
\boxed{Y(\omega_y) = \begin{cases} \frac{I}{D}X\left(\frac{I}{D}\omega_y\right) & |\omega_y| < \min(\pi, \frac{D}{I}\pi)\\ 0 & \text{otherwise} \end{cases}}
$$

Note: $\omega_y = \frac{T_y}{T_x}\omega_x = \frac{D}{I}\omega_x$.

Summary: Decimation

Example: $D = 3$

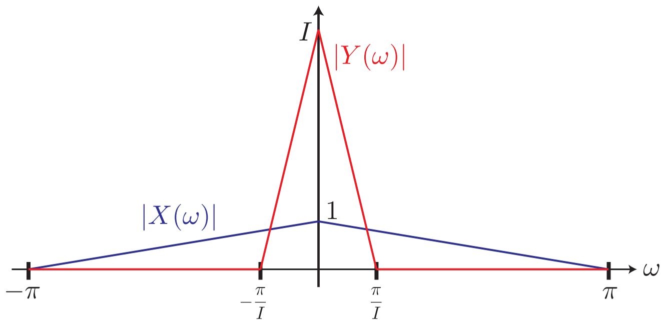
$$
\boxed{Y(\omega_y) = \frac{1}{D}X\left(\frac{\omega_y}{D}\right)}
$$



Summary: Interpolation

Example: $I = 5$

$$Y(\omega_y) = \begin{cases} I X(\omega_y I) & 0 \le |\omega_y| \le \pi/I \\ 0 & \text{otherwise} \end{cases}$$



Summary: $\frac{T_y}{T_x} = \frac{D}{I}$

Example: $D = 3, I = 5$

$$Y(\omega_y) = \begin{cases} \frac{I}{D} X\left(\frac{I}{D}\omega_y\right) & |\omega_y| < \min(\pi, \frac{D}{I}\pi) \\ 0 & \text{otherwise} \end{cases}$$