# Frequency Domain Analysis of LTI Systems

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# Discrete-Time Signals and Systems

**Reference:**

Sections 5.1, 5.2 - 5.5 of

John G. Proakis and Dimitris G. Manolakis, *Digital Signal Processing: Principles, Algorithms, and Applications*, 4th edition, 2007.





# The Frequency Response Function

- Recall for an LTI system: $y(n) = h(n) * x(n)$.
- Suppose we inject a complex exponential into the LTI system:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ x(n) &= Ae^{j\omega n} \end{aligned}$$

- Note: we consider $x(n)$ to be comprised of a pure frequency of $\omega$ rad/s





# The Frequency Response Function

$$\begin{aligned} \therefore\ y(n) &= \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega(n-k)} \\ &= \sum_{k=-\infty}^{\infty} h(k)Ae^{j\omega n} \cdot e^{-j\omega k} \\ &= Ae^{j\omega n} \cdot \underbrace{\left[\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}\right]}_{\equiv H(\omega) = DTFT\{h(n)\}} \\ &= Ae^{j\omega n}H(\omega) \end{aligned}$$

- Thus, $y(n) = H(\omega)x(n)$ when $x(n)$ is a pure frequency.

# The Frequency Response Function

$$
\begin{aligned}
y(n) &= H(\omega)x(n) \\
\text{output} &= \text{scaled input} \\
A \cdot v &= \lambda \cdot v
\end{aligned}
$$





# LTI System Eigenfunction

- Eigenfunction of a system:
  - an input signal that produces an output that differs from the input by a constant multiplicative factor
  - multiplicative factor is called the eigenvalue
- Therefore, a signal of the form $Ae^{j\omega n}$ is an eigenfunction of an LTI system.





# LTI System Eigenfunction

- Implications:
  - An LTI system can only change the amplitude and phase of a sinusoidal signal.
  - An LTI system with inputs comprised of frequencies from set $\Omega_0$ cannot produce an output signal with frequencies in the set $\Omega_0^c$ (i.e., the complement set of $\Omega_0$)





# Magnitude and Phase of $H(\omega)$

$$
H(\omega) = |H(\omega)|e^{j\Theta(\omega)}
$$

$$
\begin{aligned}
|H(\omega)| &\equiv \text{system gain for freq } \omega \\
\angle H(\omega) = \Theta(\omega) &\equiv \text{phase shift for freq } \omega
\end{aligned}
$$

# Example:

Determine the magnitude and phase of $H(\omega)$ for the three-point moving average (MA) system

$$y(n) = \frac{1}{3}\left[x(n+1) + x(n) + x(n-1)\right]$$

By inspection, $h(n) = \frac{1}{3}\delta(n+1) + \frac{1}{3}\delta(n) + \frac{1}{3}\delta(n-1)$. Therefore,

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} = \sum_{n=-1}^{1} \frac{1}{3} e^{-j\omega n} \\ &= \frac{1}{3}[e^{j\omega} + 1 + e^{-j\omega}] = \frac{1}{3}(1 + 2\cos(\omega)) \end{aligned}$$





# Example:

What is the phase of $H(\omega) = \frac{1}{3}(1 + 2\cos(\omega))$?

$$\begin{aligned} |H(\omega)| &= \frac{1}{3}|1 + 2\cos(\omega)| \\ \Theta(\omega) &= \begin{cases} 0 & 0 \le \omega \le \frac{2\pi}{3} \\ \pi & \frac{2\pi}{3} \le \omega < \pi \end{cases} \end{aligned}$$

See Figure 5.1.1 of text.





# Frequency Response of LTI Systems

| $z$-Domain | | $\omega$-Domain |
|---|---|---|
| $H(z)$ | $\overset{z=e^{j\omega}}{\Longrightarrow}$ | $H(\omega)$ |
| system function | $\overset{z=e^{j\omega}}{\Longrightarrow}$ | frequency response |
| $Y(z) = X(z)H(z)$ | $\overset{z=e^{j\omega}}{\Longrightarrow}$ | $Y(\omega) = X(\omega)H(\omega)$ |





# Frequency Response of LTI Systems

- If $H(z)$ converges on the unit circle, then we can obtain the frequency response by letting $z = e^{j\omega}$:

$$\begin{aligned} H(\omega) &= H(z)|_{z=e^{j\omega n}} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n} \\ &= \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{N} a_k e^{-j\omega k}} \end{aligned}$$

for rational system functions.

## LTI Systems as Frequency-Selective Filters

- Filter: device that discriminates, according to some attribute of the input, what passes through it
- For LTI systems, given $Y(\omega) = X(\omega)H(\omega)$
  - $H(\omega)$ acts as a kind of weighting function or spectral shaping function of the different frequency components of the signal

$$\text{LTI system} \Longleftrightarrow \text{Filter}$$





## Ideal Filters

- Classification:
  - lowpass
  - highpass
  - bandpass
  - bandstop
  - allpass

See ▸ Figure 5.4.1 of text.





## Ideal Filters

- Common characteristics:
  - unity (flat) frequency response magnitude in passband and zero frequency response in stopband
  - linear phase; for constants $C$ and $n_0$

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0} & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$





## Ideal Filters

- Suppose $H(\omega) = Ce^{-j\omega n_0}$ for all $\omega$:

$$\begin{aligned} \delta(n) &\overset{\mathcal{F}}{\longleftrightarrow} 1 \\ \delta(n-n_0) &\overset{\mathcal{F}}{\longleftrightarrow} 1 \cdot e^{-j\omega n_0} \\ C \cdot \delta(n-n_0) &\overset{\mathcal{F}}{\longleftrightarrow} C \cdot 1 \cdot e^{-j\omega n_0} = Ce^{-j\omega n_0} \end{aligned}$$

- Therefore, $h(n) = C\delta(n-n_0)$ and:

$$y(n) = x(n) * h(n) = x(n) * C\delta(n-n_0) = Cx(n-n_0)$$

# Ideal Filters

- Therefore for ideal linear phase filters:

$$H(\omega) = \begin{cases} Ce^{-j\omega n_0} & \omega_1 < |\omega| < \omega_2 \\ 0 & \text{otherwise} \end{cases}$$

  - signal components in stopband are annihilated
  - signal components in passband are shifted (and scaled by passband gain which is unity)





# Phase versus Magnitude

What's more important?



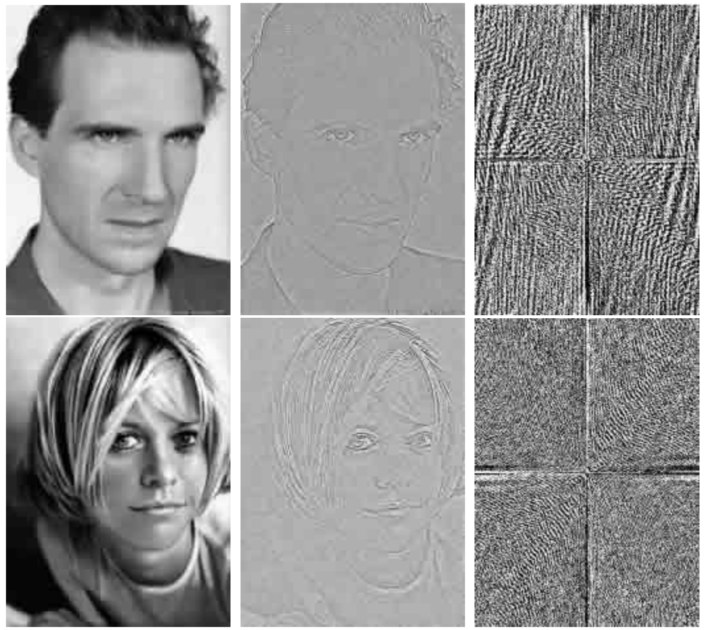






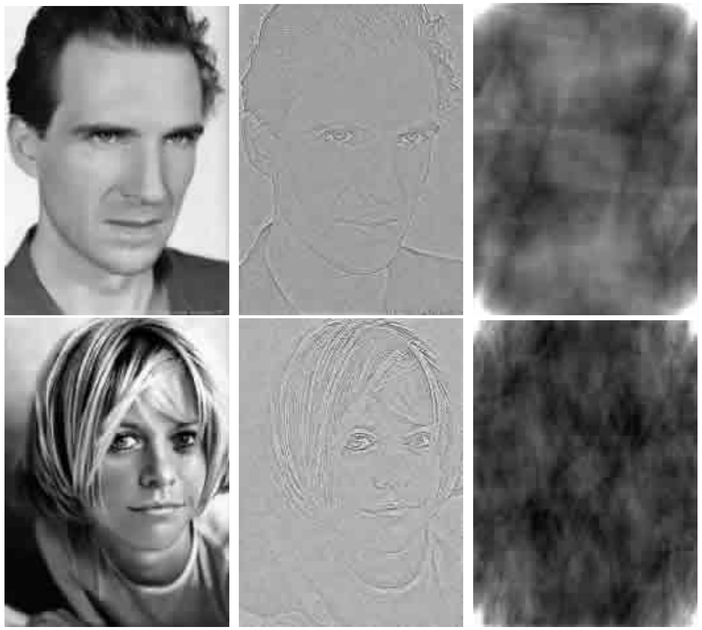
















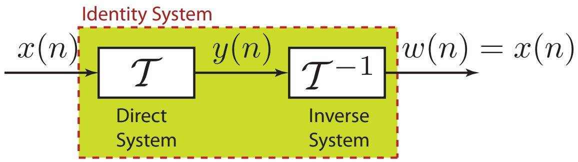
## Why Invert?

- There is a fundamental necessity in engineering applications to undo the unwanted processing of a signal.
  - reverse intersymbol interference in data symbols in telecommunications applications to improve error rate; called equalization
  - correct blurring effects in biomedical imaging applications for more accurate diagnosis; called restoration/enhancement
  - increase signal resolution in reflection seismology for improved geologic interpretation; called deconvolution

# Invertibility of Systems

- Invertible system: there is a one-to-one correspondence between its input and output signals
- the one-to-one nature allows the process of reversing the transformation between input and output; suppose

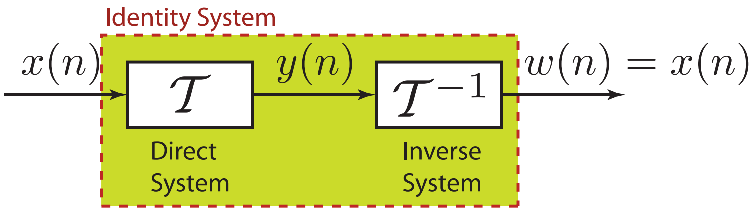
$$\begin{aligned} y(n) &= \mathcal{T}[x(n)] \quad \text{where } \mathcal{T} \text{ is one-to-one} \\ w(n) &= \mathcal{T}^{-1}[y(n)] = \mathcal{T}^{-1}\{\mathcal{T}[x(n)]\} = x(n) \end{aligned}$$

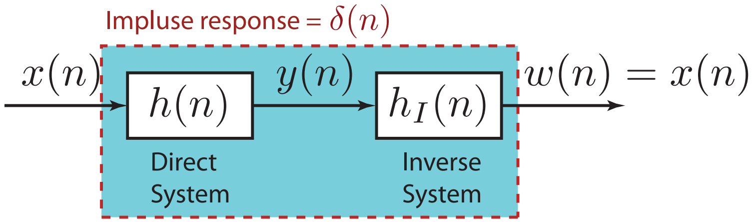







# Invertibility of LTI Systems








# Invertibility of LTI Systems

- Therefore,

$$h(n) * h_I(n) = \delta(n)$$

- For a given $h(n)$, how do we find $h_I(n)$?
- Consider the $z-$domain

$$\begin{aligned} H(z)H_I(z) &= 1 \\ H_I(z) &= \frac{1}{H(z)} \end{aligned}$$





# Invertibility of Rational LTI Systems

- Suppose, $H(z)$ is rational:

$$\begin{aligned} H(z) &= \frac{A(z)}{B(z)} \\ H_I(z) &= \frac{B(z)}{A(z)} \\ \text{poles of } H(z) &= \text{zeros of } H_I(z) \\ \text{zeros of } H(z) &= \text{poles of } H_I(z) \end{aligned}$$

# Example

Determine the inverse system of the system with impulse response $h(n) = (\frac{1}{2})^n u(n)$.





# Common Transform Pairs

| | Signal, $x(n)$ | $z$-Transform, $X(z)$ | ROC |
|---|---|---|---|
| 1 | $\delta(n)$ | $1$ | All $z$ |
| 2 | $u(n)$ | $\frac{1}{1-z^{-1}}$ | $\|z\| > 1$ |
| 3 | $a^n u(n)$ | $\frac{1}{1-az^{-1}}$ | $\|z\| > \|a\|$ |
| 4 | $na^n u(n)$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| > \|a\|$ |
| 5 | $-a^n u(-n-1)$ | $\frac{1}{1-az^{-1}}$ | $\|z\| < \|a\|$ |
| 6 | $-na^n u(-n-1)$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\|z\| < \|a\|$ |
| 7 | $(\cos(\omega_0 n))u(n)$ | $\frac{1-z^{-1}\cos\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$ | $\|z\| > 1$ |
| 8 | $(\sin(\omega_0 n))u(n)$ | $\frac{z^{-1}\sin\omega_0}{1-2z^{-1}\cos\omega_0+z^{-2}}$ | $\|z\| > 1$ |
| 9 | $(a^n \cos(\omega_0 n)u(n)$ | $\frac{1-az^{-1}\cos\omega_0}{1-2az-1\cos\omega_0+a^2z^{-2}}$ | $\|z\| > \|a\|$ |
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# Example

Determine the inverse system of the system with impulse response $h(n) = (\frac{1}{2})^n u(n)$.

- $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$, ROC: $|z| > \frac{1}{2}$; note: direct system is causal + stable.
- Therefore,
$$H_I(z) = \frac{1}{H(z)} = 1 - \frac{1}{2}z^{-1}$$
- By inspection,
$$h_I(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$
- Is the inverse system stable? causal?





# Another Example

Determine the inverse system of the system with impulse response $h(n) = \delta(n) - \frac{1}{2}\delta(n-1)$.

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=-\infty}^{\infty} [\delta(n) - \frac{1}{2}\delta(n-1)]z^{-n} \\ &= 1 - \frac{1}{2}z^{-1} \\ H_I(z) &= \frac{1}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

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# Another Example

There are two possibilities for inverses:

- Causal + stable inverse:

$$h_I(n) = \left(\frac{1}{2}\right)^n u(n)$$

- Anticausal + unstable inverse:

$$h_I(n) = -\left(\frac{1}{2}\right)^n u(-n-1)$$





# Homomorphic Deconvolution

- The complex cepstrum of a signal $x(n)$ is given by:

$$c_x(n) = Z^{-1}\{\ln(Z\{x(n)\}\} = Z^{-1}\{\ln(X(z))\} = Z^{-1}\{C_x(z)\}$$

$$x(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)$$

$$\text{cepstrum } \equiv c_x(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} C_x(z) = \ln(X(z))$$

- We say, $c_x(n)$ is produced via a homomorphic transform of $x(n)$.

# Homomorphic Deconvolution

$$
\begin{aligned}
Y(z) &= X(z)H(z) \\
C_y(z) &= \ln Y(z) \\
&= \ln X(z) + \ln H(z) \\
&= C_x(z) + C_h(z) \\
Z^{-1}\{C_y(z)\} &= Z^{-1}\{C_x(z)\} + Z^{-1}\{C_h(z)\} \\
c_y(n) &= c_x(n) + c_h(n)
\end{aligned}
$$

Therefore,

convolution in time-domain $\longleftrightarrow$ addition in cepstral domain





# Homomorphic Deconvolution

- In many applications, the characteristics of $c_x(n)$ and $c_h(n)$ are sufficiently distinct that temporal windows can be used to separate them:

$$
\begin{aligned}
\hat{c}_h(n) &= c_y(n)\hat{w}_{lp}(n) \\
\hat{c}_x(n) &= c_y(n)\hat{w}_{hp}(n)
\end{aligned}
$$

where:

$$
\begin{aligned}
\hat{w}_{lp}(n) &= \begin{cases} 1 & |n| \leq N_1 \\ 0 & |n| > N_1 \end{cases} \\
\hat{w}_{hp}(n) &= \begin{cases} 0 & |n| \leq N_1 \\ 1 & |n| > N_1 \end{cases}
\end{aligned}
$$





# Homomorphic Deconvolution

- Obtaining the inverse homomorphic transforms of $c_h(n)$ and $c_x(n)$ give estimates of $h(n)$ and $x(n)$, respectively.

$\blacksquare$